Price Changes and Welfare Analysis: Measurement under Individual Heterogeneity

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Abstract

Measuring the welfare impact of price changes on consumers is pivotal in economic analyses. Researchers often measure these impacts with cross-sectional data, where every consumer is observed only once. The representative agent (RA) approach, which assumes all observations stem from a single agent, may lead to biased estimates when agents’ preferences are heterogeneous. We show how to use the higher moments of demand to improve these estimates. In fact, the variance alone captures much of the bias in the RA approach. Our approach also enables inference on the distribution of welfare changes. We then leverage our approach to obtain conditions moments of demand must satisfy to arise from a population of utility maximizers and deliver a characterization of rationality for the two-good case. Using the UK Household Budget Survey, we apply our methodology to estimate the welfare impact of a 10% transport price increase and find that the RA approach understates the welfare impact by 27.2%.

Keywords: nonparametric welfare analysis, individual heterogeneity, compensating variation, exact consumer surplus, deadweight loss

JEL classification: C14, C31, D11, D12, D63, H22, I31

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1 Introduction

Measuring the welfare impact of price changes on consumers is crucial in many settings, for example, for policy evaluation of tax reforms and trade liberalization. The ideal way to estimate this impact would be a long panel on individuals’ consumption choices. Long panel data can be used to estimate individual demand functions and preferences and identify the welfare impact. However, the data typically available to measure such impacts takes the form of cross-sections, meaning it is only possible to observe one consumption bundle for every consumer. This presents a theoretical challenge since only a single point of individuals’ entire demand function is observed.

The standard representative agent (RA) approach assumes that all observations come from the same individual (e.g., see Hausman, 1981; Vartia, 1983). Under this assumption, the average demand function coincides with the RA. This approach works well if the data comes from individuals with similar preferences. However, it might misstate welfare impacts when there is considerable preference heterogeneity.

We present a method to improve upon the RA approach when only cross-sectional data is available. We make a methodological contribution by deriving the relationship between the conditional moments of demand and the Slutsky equation. This allows us to use the information in cross-sectional data about income effects. Knowledge of income effects can be used for welfare calculations, as they show how much individuals need to be compensated for a price change.

The techniques developed in this paper can also be used to test stochastic rationalizability. We can characterize rationalizable cross-sectional distributions of demand locally in the two-good case. Our method is computationally feasible, and we use it to construct a semi-decidable test of rationality. With more than two goods, we can test negative semidefiniteness of compensated demand but not symmetry.

Returning to the analysis of welfare impacts, we show that one problem with the RA approach is that it weights all individuals’ marginal propensity to consume equally. Consider two individuals who have different demands and different marginal propensities to consume. Correctly computing the welfare impact involves weighting each individual’s marginal propensity by the amount they demand. Thus, the equal weighting in the RA

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1 Note that for a single individual, the demand function pins down the utility function, making such computations possible (Hurwicz and Uzawa, 1971).

2 Allowing for heterogeneous preferences is essential in empirical applications since traditional microeconometric models typically only explain a small part of the variation in consumer demand. Moreover, interpreting average demand as a representative consumer is only justified under restrictive assumptions (Jerison, 1994; Lewbel, 2001).

3 The moments of demand are conditioned on prices and income.

4 See McFadden and Richter (1991) and McFadden (2005) for a thorough treatment of stochastic rationalizability. Perhaps surprisingly, we find that even just the first and second moments of demand can be used to construct tests of rationality and provide tight estimates of average welfare changes. By contrast, the first conditional moment of demand carries no empirical content locally (for a detailed review, see Rizvi, 2006).
approach may produce substantial bias in the welfare impact.

In practice, we find that a large amount of the bias incurred by the RA approach is captured by the covariance between the amount demanded and the marginal propensity to consume, which allows us to (locally) re-weight the marginal propensities. Under standard assumptions, this covariance can be estimated using the variance of demand, which can, in turn, be estimated from cross-sectional data. It delivers the average direction of the bias caused by the RA approach. In other words, the RA approach assumes this covariance to be zero. We show that our bias correction can improve welfare estimates significantly, especially when the analyst has no accurate a priori knowledge of the magnitude of the income effects. Besides average welfare, our results also enable inference on the distribution of welfare under general forms of preference heterogeneity. This enables a distributional assessment of the impact of price changes.

To demonstrate our approach’s usefulness, we estimate the welfare effect of a 10% increase in transport prices on consumer welfare. We collect data on households’ consumption bundles and income from 14 waves of the UK Household Budget Survey (2006-2019), and on prices from the Office for National Statistics. Our results suggest that the RA approach significantly underestimates the welfare impact, as our estimate of the welfare impact is 27.2% higher. Moreover, this bias is larger for individuals with low disposable income. We perform a similar exercise for food and housing and find even larger effects for the latter.

**Related Literature.** The literature on the RA approach is first found in Hausman (1981) and Vartia (1983). Hausman and Newey (1995) obtain point estimates for a representative consumer using nonparametric regression. Foster and Hahn (2000) and Blundell, Browning, and Crawford (2003) give conditions under which these point estimates are first-order approximations to the true welfare impact.

More recently, Hausman and Newey (2016) show that the average welfare impact is not point-identified from cross-sectional data. However, they demonstrate that if income effects are bounded, observationally equivalent models’ average welfare estimates are close. They show how to compute worst-case bounds on these effects. We strengthen their insight that this non-identification result generally has limited empirical consequence for welfare analysis by providing the best possible point estimates and tightening their bounds.

Several papers attempt to provide bounds that account for preference heterogeneity in this tradition. Making stronger assumptions on preferences, Schlee (2007) shows that the estimates from the RA approach can act as an upper bound for the true value. Other papers provide bounds by means of revealed preference inequalities. Exploiting the weak

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5While their method is robust, it can lead to wide bounds when the analyst has no prior knowledge of the magnitude of the income effect.
axiom of stochastic revealed preference, Cosaert and Demuynck (2018) derive bounds for a sample of heterogeneous consumers observed repeatedly. Kitamura and Stoye (2019) carry out a similar analysis in the case of random utilities. Chambers and Echenique (2021) provide bounds by characterizing allocations which cannot be rejected as Pareto optimal. Kang and Vasserman (2022) study settings where only a few aggregate demand bundles are observed and assess the additional power that assumptions relating to the curvature of demand provide. These bounding approaches typically deliver wide bounds, which might limit their usefulness for policy analysis.

Hoderlein and Vanhems (2018) deliver point estimates under the assumption that demand is monotonic in scalar unobserved heterogeneity. The identifying assumption is restrictive, however, as it implies that the relative position of an individual in the conditional distribution of demand is unchanged when prices or income change. This is unrealistic when the marginal propensity to consume varies widely across individuals. Moreover, their results are only applicable to settings with two goods.

In the discrete choice literature, Dagsvik and Karlström (2005), de Palma and Kilani (2011), and Bhattacharya (2015, 2018) show that the distribution of the compensating variation can be written in terms of choice probabilities. These choice probabilities are point-identified from cross-sectional data, even when heterogeneity is unrestricted.6

Our results on stochastic rationalizability are related to the literature that derives observable restrictions on demand. In the many-good case, Hoderlein and Stoye (2014, 2015) and Dette, Hoderlein, and Neumeyer (2016) derive and test restrictions on marginal quantiles of demand. In a related exercise, Hoderlein (2011) uses techniques similar to ours to bound the proportion of individuals in a population who could satisfy rationality. An advantage of our approach is that it can also be employed when researchers do not observe the entire demand distribution but only some coarse moments.7 Kitamura and Stoye (2018) provide tests based on revealed preference inequalities for finitely many demand distributions at different prices. By contrast, our results assume differentiable demands but are valid at the population level.

We also view our results as challenging the intuition behind the Sonnenschein-Mantel-Debreu theorem (Sonnenschein, 1973; Mantel, 1974; Debreu, 1974), which suggests that rationality imposes no restrictions on aggregate demand.8 In the specific case addressed by Chiappori and Ekeland (1999), where the authors fix nominal incomes and vary prices, we find that rationality imposes restrictions on higher demand moments.9 Importantly,

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6However, if choice is ordered, identification breaks down due to a lack of relative price variation. Since continuous choice under a linear budget constraint can be seen as a limiting case of ordered discrete choice, this finding is consistent with the non-identification result in Hausman and Newey (2016).

7Moreover, our moment-based results scale naturally to the many-good case, whereas the quantile-based approach does not.

8For a review of the literature on the restrictions rationality places on demand functions see Chiappori and Ekeland (2011).

9Hildenbrand (1983, 1994) and Härdle, Hildenbrand, and Jerison (1991) impose restrictions on the
we show that just the first two moments of demand already contain empirical content.

## 2 Illustrative Example

The intent of this paper is to measure the welfare impact of a price change, specifically the compensating variation (CV). Assume that preferences in a population are indexed by $\omega \in \Omega$, and drawn from a distribution $F$. A researcher observes the consumption bundles $(q_i)$ of a sample of individuals $i = 1, \ldots, k$, and the prices of the goods they buy $(p_i)$. The researcher observes each individual’s choices only once. Note that the researcher can infer individuals’ income levels from their consumption and the prices they face ($y_i = q_i \cdot p_i$). We illustrate the intuition behind our results by considering the setting where both uncompensated and compensated demand are linear in price.

**Individual Welfare.** Let $q^\omega(p, y)$ denote uncompensated demand and $h^\omega(p, u)$ compensated demand for a given type $\omega$. $u$ denotes utility. Assume that both demands are linear in price. The welfare impact of a price change from $p_0$ to $p_1$ at income $y$ is measured by the CV:

$$CV^\omega(p_0, p_1, y) = \int_{p_0}^{p_1} h^\omega(p, u_i) dp.$$  

Because $h(p, u)$ is linear, we can rewrite CV as follows:

$$CV^\omega(p_0, p_1, y) = \int_{p_0}^{p_1} \left[ h^\omega(p_0, u) + (p - p_0) \frac{\partial h^\omega(p_0, u)}{\partial p} \right] dp = \Delta p h^\omega(p_0, u) + \frac{(\Delta p)^2}{2} \frac{\partial h^\omega(p_0, u)}{\partial p} \quad (1)$$  

where $\Delta p = p_1 - p_0$. Observe that $h^\omega(p_0, u) = q^\omega(p_0, y)$. Moreover, Slutsky’s equation allows us to decompose the change in compensated demand into two terms, the substitution effect and the income effect (IE):

$$\frac{\partial h^\omega(p, u)}{\partial p} = \frac{\partial q^\omega(p, y)}{\partial p} + q^\omega(p, y) \frac{\partial q^\omega(p, y)}{\partial y}.$$  

Hence we can rewrite 1 as follows:

$$CV^\omega(p_0, p_1, y) = \Delta p q^\omega(p_0, y) + \frac{(\Delta p)^2}{2} \left[ \frac{\partial q^\omega(p, y)}{\partial p} + q^\omega(p_0, y) \frac{\partial q^\omega(p, y)}{\partial y} \right]. \quad (3)$$  

The variance of demand (second moment) which guarantees that aggregate demand obeys the so-called law of demand. We tackle the inverse problem.
The RA Approach. We assume the analyst has a large enough sample to estimate 
\( M_1(p, y) = \mathbb{E}_\omega[q^\omega \mid p, y] \) as a function of price and income. Suppose that there is a single preference type \( \omega^* \). Their demand \( q^{\omega^*}(p, y) \) would equal \( M_1(p, y) \). Making this assumption lets us compute the hypothetical welfare change \( CV^{\omega^*}(p_0, p_1, y) \) from Expression (3):

\[
\mathbb{E}[CV^{\omega^*}(p_0, p_1, y)] = CV^{\omega^*}(p_0, p_1, y) = \Delta p M_1 + \left( \frac{\partial M_1}{\partial p} + M_1 \frac{\partial M_1}{\partial y} \right) \tag{4}
\]

The RA approach assumes there is a representative agent \( \omega^* \) and uses 4 to estimate the welfare impact of a price change.\(^{10}\)

Correcting for Preference Heterogeneity. The issue with the RA approach is that it fails to account for preference heterogeneity. However, individuals may have large unobserved preference heterogeneity at every price and income level in the data. As we will see, if there is significant heterogeneity in a population’s rates of substitution, the RA approach may lead to significantly biased estimates. Lemma 1, which follows from Theorem 1 in the next section, quantifies this bias.

Lemma 1. In the linear case, the bias in the RA approach is:

\[
CV^* - CV_{RA} = \frac{(\Delta p)^2}{2} \text{Cov} \left( q^{\omega^*}, \frac{\partial q^{\omega^*}}{\partial y} \right).
\]

The bias in the RA approach is proportional to the covariance between demand and marginal propensity to consume. The example which follows will deliver intuition for this result. The RA approach implicitly assumes this covariance is zero; accounting for this covariance can considerably improve welfare estimates.\(^{11}\)

Example. To understand this bias, consider a setting with two individuals who have different preferences: Ann (A) and Betty (B). Let their demands be \( q^A, q^B \), which would fix \( M_1 = \frac{q^A + q^B}{2} \). The bias in the RA approach stems from the misspecification of income effects in equation 4. In the RA approach, the analyst estimates the average income effect assuming it stems from one individual, which gives

\[
\overline{IE}_{RA} = M_1 \frac{\partial M_1}{\partial y} = \frac{1}{2} \left( \overline{q} \frac{\partial q^A}{\partial y} + \overline{q} \frac{\partial q^B}{\partial y} \right).
\]

\(^{10}\)In this approach, the variation in demand conditional on prices and income is thought to be generated by measurement error. See, for example, Hausman and Newey (1995).

\(^{11}\)Note that demand conditional on prices and income is a degenerate random variable in the RA approach.
However, if we conducted the analysis individual by individual, the average income effect would be

\[ \bar{IE}^* = \frac{1}{2} \left( IE^A + IE^B \right) = \frac{1}{2} \left( q^A \frac{\partial q^A}{\partial y} + q^B \frac{\partial q^B}{\partial y} \right) , \]

Note that the difference stems from how the income derivative \( \frac{\partial q}{\partial y} \) is weighted across individuals. Table 1 summarizes these weights for the two approaches.

<table>
<thead>
<tr>
<th></th>
<th>Ann</th>
<th>Betty</th>
</tr>
</thead>
<tbody>
<tr>
<td>RA weights</td>
<td>( q^A )</td>
<td>( q^B )</td>
</tr>
<tr>
<td>Actual weights</td>
<td>( q^A )</td>
<td>( q^B )</td>
</tr>
</tbody>
</table>

Now suppose that Anne and Betty’s demands are such that \( q^A < q^B \) and \( \frac{\partial q^A}{\partial y} < \frac{\partial q^B}{\partial y} \), which means that \( \text{Cov}(q, \frac{\partial q}{\partial y}) > 0 \). Since \( q^A < q < q^B \), in the RA approach, the weight on Betty is too small and the weight on Ann too large. As \( \frac{\partial q^A}{\partial y} < \frac{\partial q^B}{\partial y} \), there is too much weight on small values of \( \frac{\partial q}{\partial y} \), which biases the RA approach downwards.\(^\text{12}\)

## 3 Conceptual Framework

Our conceptual framework allows for unobserved heterogeneity in preferences to be unrestricted. For ease of exposition, we suppress all observed individual characteristics; all results in this paper can be thought of as conditional on these covariates.

### 3.1 Consumer Demand

We consider the standard model of utility maximization under a linear budget constraint. Let \( \Omega \) denote the universe of preference types. Every preference type \( \omega \in \Omega \) can be considered an individual with preferences over bundles of \( (k+1) \) goods \( q \). We assume the set of bundles is compact and convex and denote it as \( Q \subseteq \mathbb{R}^{k+1}_+ \). Preferences are assumed to be representable by smooth, strictly quasi-concave utility functions \( u^\omega : Q \to \mathbb{R} \). This formulation allows utility functions to differ arbitrarily across individuals. Prices are denoted \( p \in P \subset \mathbb{R}^{k+1}_+ \) and income, \( y \in Y \subset \mathbb{R}_+ \). We call a pair \((p, y)\) a budget set.

Individual demand functions \( q^\omega(p, y) : P \times Y \to Q \) arise from individuals maximizing their utility subject to a linear budget constraint,

\[ q^\omega(p, y) = \arg \max_{p \cdot q \leq y} u^\omega(q). \]

\(^\text{12}\)In the odd case where \( \text{Cov}(q, \frac{\partial q}{\partial y}) < 0 \), the RA approach is biased upwards.
These demand functions satisfy homogeneity of degree zero and Walras law,

\[ q^\omega(\alpha p, \alpha y) = q^\omega(p, y), \quad \forall \alpha \in \mathbb{R}_+, \]
\[ p \cdot q^\omega(p, y) = y, \]

for all budget sets. For every uncompensated (Marshallian) demand function \( q^\omega \), there exists a compensated (Hicksian) demand function \( h^\omega(p, u) : \mathcal{P} \times \mathbb{R} \to \mathcal{Q} \) defined as

\[ h^\omega(p, u) = \arg \min_{q \in \mathcal{Q}} \{ p \cdot q | u^\omega(q) \geq u \}. \]

The Slutsky equation (5) provides a link between both demand functions.

\[ \frac{\partial}{\partial p} q^\omega(p, y) = \frac{\partial}{\partial p} h^\omega(p, u) - \frac{\partial}{\partial y} q^\omega(p, y) q^\omega(p, y)^T, \quad (5) \]

The indirect utility function \( v^\omega : \mathcal{P} \times \mathcal{Y} \to \mathbb{R} \) is defined as

\[ v^\omega(p, y) = \max_{p \cdot q \leq y : q \in \mathcal{Q}} u^\omega(q), \]

i.e., the utility level obtained at budget set \((p, y)\). The expenditure function \( e^\omega(p, u) : \mathcal{P} \times \mathbb{R} \to \mathcal{Y} \) is defined as

\[ e^\omega(p, u) = \min_{u \leq u^\omega(q) : q \in \mathcal{Q}} p \cdot q, \]

i.e., the minimum amount of income needed to achieve utility level \( u \) at prices \( p \). These two functions are related by Shephard’s Lemma (6).

\[ \frac{\partial}{\partial p} e^\omega(p, u) = h^\omega(p, u). \quad (6) \]

In the remainder of the paper, we will omit the demand and price for the \((k + 1)\)st good using Walras’ law.

We assume that preference types are distributed with some distribution \( F(\omega) \), which admits a density. We now state our main identifying assumption.

**Assumption 1.** The distribution of unobserved heterogeneity is independent of prices and income:

\[ F(\omega | p, y) = F(\omega). \]

The exogeneity of budget sets is a strong but standard assumption in the literature on nonparametric identification. (e.g., see Hausman and Newey, 2016; Blomquist, Newey, Kumar, and Liang, 2021). To the best of our knowledge, theoretical results for cross-sections do not allow for general forms of endogeneity under general heterogeneity. Some
forms of endogeneity can be mitigated by a control function approach (Blundell and Powell, 2003).

3.2 Welfare Impact

Our main object of interest is the compensating variation (CV), which quantifies the impact of price changes on individual welfare.\(^{13}\) It measures how much income an individual is willing to give up after the price change to be offered the initial price vector. Formally, for a price change from \(p_0\) to \(p_1\), it is defined as

\[
CV^\omega(p_0, p_1, y) = e^\omega(p_1, v^\omega(p_0, y)) - e^\omega(p_1, v^\omega(p_1, y)) - y.
\]

When \(p_1 > p_0\), we have that \(CV^\omega(p_0, p_1, y) > 0.\(^{14}\) Notice that the compensating variation is stochastic from the analyst’s viewpoint because individuals’ preference types cannot be observed. We let \(\Delta p = p_1 - p_0\).

3.3 Conditional Moments of Demand

In the two-good case, integrating out unobserved preference heterogeneity, we can express the \(n\)th (non-central) conditional moment of demand as

\[
M_n(p, y) = \mathbb{E}_\omega[q^\omega(p, y)^n | p, y] = \int q^\omega(p, y)^n dF(\omega | p, y) = \int q^\omega(p, y)^n dF(\omega),
\]

since, by Walras’ law, it suffices to consider scalar demand.\(^{15}\) The second equality follows from Assumption 1. Since these moments are conditional expectation functions, the set \(\{M_n(p, y)\}_{n=1}^\infty\) is nonparametrically identified from cross-sectional data.

In the many-good case, one can express the \(n\)th conditional moment of demand by means of the symmetric \(n\) tensor \(T^\omega_n(p, y)\) for which the element \(t^\omega_{i_1, i_2, \ldots, i_n}(p, y) = q^\omega_{i_1}(p, y)q^\omega_{i_2}(p, y) \cdots q^\omega_{i_n}(p, y)\) with \(i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, k\}\). We define the generalized

\(^{13}\)We focus on compensating variation (instead of equivalent variation) because this measure allows comparing different reforms and is measured in baseline prices.

\(^{14}\)For expository clarity of our results, we deviate from the textbook definition of the CV by reversing its sign (e.g., see Mas-Colell, Whinston, and Green, 1995).

\(^{15}\)In the remainder of the paper, unless stated otherwise, expectations are always conditional on a budget set \((p, y)\): i.e., for a random variable \(z(p, y)\), we will write \(\mathbb{E}[z(p, y)] = \mathbb{E}[z(p, y) | p, y]\).
tensor form of $T_n(\omega, p, y)$ with respect to a vector $v \in \mathbb{R}^k$ as the multilinear function

$$v(**)T_n(\omega, p, y) = T_n(p, y)(v \times v \cdots \times v)$$

$$= \sum_{i_1, i_2, \ldots, i_n} t_{i_1, i_2, \ldots, i_n}(p, y) v_{i_1} v_{i_2} \cdots v_{i_n}.$$

Again, by integrating out unobserved preference heterogeneity, we can express the $n$th (non-central) conditional moment of demand as

$$M_n(p, y) = \mathbb{E}[T_n(\omega, p, y)] = \int T_n(p, y) dF(\omega | p, y). \tag{8}$$

We define a moment sequence as the (possibly infinite) sequence $\{M_i(p, y)\}_{i=1}^n$ of the first $n$ moments of demand.

### 3.4 Rationalizability

Let $\{a_i(p, y)\}_{i=1}^r$ be a sequence where each $a_i(p, y) : \mathcal{P} \times \mathcal{Y} \rightarrow \mathbb{R}^{(k)^i}$ is function which maps budget sets to tensor forms of (weakly) increasing dimension. We say $\{a_i(p, y)\}_{i=1}^r$ is rationalizable around $(p_0, y_0)$ if there exists a universe of preference types $\Omega$ and a probability measure $F(\omega)$ over these types such that

$$a_i(p, y) = \int T_i(\omega, p, y) dF(\omega), \quad \forall i \leq k,$$

holds for an open set around the budget set $(p_0, y_0)$, and $T_i(\omega)$ is generated by a rational demand function $q_\omega$ for all $\omega \in \Omega$.

Technical conditions are relegated to Appendix A. In particular, we assume that the conditions for the dominated convergence theorem hold such that derivative and integral operators can be interchanged.

### 4 Approximations to Welfare Changes

We now formalize and extend the procedure that underpins our illustrative example. For ease of exposition, we focus on the two-good case; the results for the many-good case are relegated to Appendix B.

In Section 4.1, we derive results for small price changes, where “triangles are good approximations.” This allows us to obtain a first-order approximation of compensated

\[\text{A demand function is called rational when it obeys Slutsky symmetry and negative semidefiniteness, homogeneity of degree zero, and Walras’ law.}\]
demand in terms of observable objects, which then lets us derive a second-order approximation to all moments of the CV. In addition, we show that cross-sectional data is uninformative about higher-order approximations. Figure 1 gives a schematic overview of our main argument.

In Section 4.2, we derive results for settings where price changes can be large, requiring us to move away from triangles. We allow demand to vary non-linearly in prices, but it remains linear in income. We demonstrate that a second-order approximation to the average CV is identified from cross-sectional data, but higher-order moments are not.

Figure 1: Schematic overview of our main argument

### 4.1 Linearity in Price and Income

We show that the moments of CV can be approximated up to second order from the conditional moments of demand. The following lemma establishes a relation between (transformations of) income effects and the conditional moments of demand.\(^{17}\)

**Lemma 2.** For every \(n \in \mathbb{N}_+\), it holds that

\[
E \left[ \left( q^{\omega}(p, y) \right)^{n-1} \frac{\partial}{\partial y} q^{\omega}(p, y) \right] = \frac{1}{n} \frac{\partial}{\partial y} M_n(p, y).
\]

**Proof.** Using the definition of the conditional moments in Expression (7), we know that

\[
\frac{\partial}{\partial y} M_n(p, y) = \frac{\partial}{\partial y} \left( \int q^{\omega}(p, y)^n dF(\omega) \right).
\]

\(^{17}\)A full exploration of the informational content of the moments of demand is postponed to Section 5.
Interchanging the derivative and integral operators gives us

\[
\frac{\partial}{\partial y} M_n(p, y) = \int \frac{\partial}{\partial y} q^\omega(p, y)^n dF(\omega) \\
= n \int q^\omega(p, y)^{n-1} \frac{\partial}{\partial y} q^\omega(p, y) dF(\omega) \\
= n \mathbb{E} \left[ q^\omega(p, y)^{n-1} \frac{\partial}{\partial y} q^\omega(p, y) \right].
\]

\[\square\]

We now use our knowledge of income effects to compute a linear approximation to compensated demand. We then appeal to Shephard’s lemma to calculate changes in the expenditures via integrating compensated demand. This yields a second-order approximation to the EV. This is summarized in the following theorem.

**Theorem 1.** The second-order approximation of the nth moment of the CV depends only on the nth and \((n + 1)\)st conditional moment of demand.\(^{18}\)

\[
\mathbb{E}[CV^\omega(p_0, p_1, y)^n] = (\Delta p)^n \left( M_n(p_0, y) + \frac{\Delta p}{2} \left[ \frac{\partial M_n(p_0, y)}{\partial p} + \frac{n}{n + 1} \frac{\partial M_{n+1}(p_0, y)}{\partial y} \right] + O((\Delta p)^2) \right).
\]

**Proof.** We only consider the case for the average CV for clarity of exposition. For the other moments, refer to Appendix C. Observe that by Shephard’s lemma (6),

\[
\frac{\partial}{\partial p} e^\omega(p, u) = h^\omega(p, u),
\]

so that we can write the CV in terms of compensated demand,

\[
CV^\omega(p_0, p_1, y) = \int_0^1 h^\omega(p(t), v^\omega(p_0, y)) dp,
\]

for some continuous price path \(p: [0, 1] \to \mathcal{P}\) with \(p(0) = p_0\) and \(p(1) = p_1\). Without loss of generality, we will assume the price path to be linear, i.e., \(p(t) = p_0 + t \Delta p\), such that\(^{19}\)

\[
CV^\omega(p_0, p_1, y) = \Delta p \int_0^1 h^\omega(p_0 + t \Delta p, v^\omega_0) dt,
\]

and therefore

\[
\mathbb{E}[CV^\omega(p_0, p_1, y)] = \Delta p \int_0^1 \mathbb{E}[h^\omega(p_0 + t \Delta p, v^\omega_0)] dt, \tag{9}
\]

where \(v^\omega_0 = v^\omega(p_0, y)\).

\(^{18}\)We let \(O\) denote Landau’s big \(O\).

\(^{19}\)The integral is path independent due to Slutsky symmetry.
We now combine the Slutsky equation (5) and Lemma 2 to derive the expectation of the price derivative of compensated demand. In particular, we have that

\[ E \left[ \frac{\partial}{\partial p} h_\omega(p, y) \right] = E \left[ \frac{\partial}{\partial p} q_\omega(p, y) + q_\omega(p, y) \frac{\partial}{\partial y} q_\omega(p, y) \right] = \frac{\partial}{\partial p} M_1(p, y) + \frac{1}{2} \frac{\partial}{\partial p} M_2(p, y), \]

for every \((p, y) \in \mathcal{P} \times \mathcal{Y}\). This allows us to derive a first-order approximation to average compensated demand around \(t = 0\):

\[ E \left[ h_\omega(p_0 + t \Delta p, v_\omega^0) \right] = M_1(p_0, y) + t \Delta p \left( \frac{\partial}{\partial p} M_1(p_0, y) + \frac{1}{2} \frac{\partial}{\partial p} M_2(p_0, y) \right) + O((\Delta p)^2). \]

Plugging this approximation into Expression (9) gives us

\[ E\left[ CV_\omega(p_0, p_1, y) \right] = \Delta p \int_0^1 \left[ M_1(p_0, y) + t \Delta p \left( \frac{\partial}{\partial p} M_1(p_0, y) + \frac{1}{2} \frac{\partial}{\partial p} M_2(p_0, y) \right) + O((\Delta p)^2) \right] dt = \Delta p M_1(p_0, y) + \frac{(\Delta p)^2}{2} \left( \frac{\partial}{\partial p} M_1(p_0, y) + \frac{1}{2} \frac{\partial}{\partial p} M_2(p_0, y) \right) + O((\Delta p)^3). \]

Specifically, the second-order approximation to the average CV only uses information from the conditional mean and the variance, the first two conditional moments of demand.

This theorem informs us that the first two terms of the series expansion for all moments of the compensating variation can be identified in the neighbourhood of a budget set. In the following theorem, we show that, in some sense, this is the best approximation that can be obtained from cross-sectional data.

**Theorem 2.** The \(k\)th-order approximation of the \(n\)th moment of the CV for \(k \geq 3\) is not identified from the conditional moments of demand.

**Proof.** For clarity of exposition, we only consider the case for the average CV and \(k = 3\). Suppose the true series expansion of the CV at some budget set \((p_1, y)\) can be written as

\[ E[CV_\omega(p_0, p_1, y)] = a_0 + a_1 \Delta p + a_2 (\Delta p)^2 + a_3 (\Delta p)^3. \]

By extending the argument in the proof of Theorem 1, to recover \(a_3\), one must identify \(E \left[ D^2_p h_\omega(p, v_\omega^0) \right] \), i.e., the expected second price derivative of compensated demand. By differentiating the identity \(h_\omega(p, u) = q_\omega(p, e_\omega(p, u))\) twice with respect to price, taking expectations, and interchanging differentiation and integration, one obtains that

\[ E[D^2_p h_\omega(p, v_\omega^0)] = D^2_p M_1(p_0, y) + \frac{1}{2} D_{p,y} M_2(p_0, y) + \frac{1}{3} D^2_y M_3(p_0, y) - E \left[ q_\omega(p_0, y) \left( \frac{\partial}{\partial y} q_\omega(p_0, y) \right)^2 \right]. \]
As a direct consequence of Lemma 4 in the Appendix, the final term cannot be identified from cross-sectional data. That is, we show that two observationally equivalent models can generate different values for $E[q^\omega(p_0, y)(\frac{\partial}{\partial y}q^\omega(p_0, y))^2]$. This implies that the third-order approximation of the average CV is also not identified.

\[ E \left[ q^\omega(p, y) \left( \frac{\partial}{\partial y}q^\omega(p, y) \right)^2 \right] = E \left[ q^\omega(p, y) \left( \frac{\partial}{\partial y}q^\omega(p, y) \right)^2 \mid q^\omega(p, y) \right]. \]

This highlights that $E[q^\omega(p, y)(\frac{\partial}{\partial y}q^\omega(p, y))^2]$ is equal to the (non-centered) covariance between the demand bundle and the second moment of the income effect at that demand bundle. Therefore, failure to identify the third-order approximation of average welfare is due to cross-sectional data being uninformative about how the variance of the income effect varies across demand bundles.

Remark 1. Akin to Hausman (1981), if the price of only one good changes, only knowledge of the good’s demand is needed, reducing the analysis from many goods to two.\(^{20}\)

Remark 2. It is no coincidence that $E[q^\omega(p, y)(\frac{\partial}{\partial y}q^\omega(p, y))^2]$ is not identified from cross-sectional data. Using the law of iterated expectations, we can write

\[ E \left[ q^\omega(p, y) \left( \frac{\partial}{\partial y}q^\omega(p, y) \right)^2 \right] = E \left[ q^\omega(p, y) \left( \frac{\partial}{\partial y}q^\omega(p, y) \right)^2 \mid q^\omega(p, y) \right]. \]

Direct application of Theorem 2.1 in Hoderlein and Mammen (2007) shows that in nonseparable models, cross-sectional data identifies local average structural derivatives (e.g., $E[\frac{\partial}{\partial y}q^\omega(p, y) \mid q^\omega(p, y)]$) but not transformations of these local average structural derivatives (e.g., $E[(\frac{\partial}{\partial y}q^\omega(p, y))^2 \mid q^\omega(p, y)]$). This is why $E[D_p h^\omega(p, v^\omega_1)]$ is identified, but $E[D_p^2 h^\omega(p, v^\omega_1)]$ is not. The same reasoning holds for the higher-order approximations, mutatis mutandis.

Remark 3. Since cross-sectional data identifies the local average structural derivatives $E \left[ \frac{\partial}{\partial y}q^\omega(p, y) \mid q^\omega(p, y) = \overline{q} \right]$ and $E \left[ \frac{\partial}{\partial y}q^\omega(p, y) \mid q^\omega(p, y) = \overline{q} \right]$, the approximation for the average CV developed in Theorem 1 could be made conditional on a given demand bundle $\overline{q}$. Formally, we have that

\[ E \left[ \frac{\partial}{\partial p} h^\omega(p_0, y) \mid q^\omega(p_0, y) = \overline{q} \right] = E \left[ \frac{\partial}{\partial p} q^\omega(p_0, y) + q^\omega(p_0, y) \frac{\partial}{\partial y} q^\omega(p_0, y) \mid q^\omega(p_0, y) = \overline{q} \right] = E \left[ \frac{\partial}{\partial p} q^\omega(p_0, y) \mid q^\omega(p_0, y) = \overline{q} \right] + \overline{q} E \left[ \frac{\partial}{\partial y} q^\omega(p_0, y) \mid q^\omega(p_0, y) = \overline{q} \right], \]

where the RHS is identified. This expression shows that our method could be used to improve welfare estimates bundle by bundle if the entire demand model could be nonparametrically estimated. In practice this may be very demanding on the data.

\(^{20}\)See Appendix B.
Remark 4. Information on the income effects can also be used to construct informative bounds on changes in welfare. By the mean value theorem,

\[ E[CV^\omega(p_0, p_1, y)] = \Delta p M_1(p_0, y) + \frac{(\Delta p)^2}{2} \left( \frac{\partial}{\partial p} M_1(p_0, y) + \frac{1}{2} \frac{\partial}{\partial y} M_2(p_0, y) \right) + \frac{(\Delta p)^3}{6} D_p^2 E[h^\omega(p, y)], \]

for some intermediate price \( \bar{p} \in [p_0, p_1] \).

If \( \Delta p > 0 \) and if compensated demand is convex in prices, the second-order approximation yields a lower bound. \(^{21}\)

On the other hand, if \( \Delta p < 0 \) and if compensated demand is convex in prices, our approximation acts as an upper bound. When the good is also normal, we have that

\[ CV^\omega(p_0, p_1, y) \geq CS^\omega(p_0, p_1, y), \]

such that

\[ E[CV^\omega(p_0, p_1, y)] \geq E[CS^\omega(p_0, p_1, y)], \]

where \( CS^\omega(p_0, p_1, y) = \Delta p \int_0^1 q^\omega(p(t), y) dt \). Therefore, one can obtain two-sided bounds in this case. The above remark demonstrates that our approach can be leveraged beyond just approximations, specifically, to construct bounds.

Remark 5. Using a similar insight to that in Lemma 2 it is possible to calculate average income elasticities nonparametrically. Let \( \eta^\omega(p, y) \) denote an individual’s income elasticity for the budget set \( p, y \). It holds that

\[ E[\eta^\omega(p, y)] = E \left[ \frac{y}{q^\omega(p, y)} \frac{\partial}{\partial y} q^\omega(p, y) \right] = y E \left[ \frac{1}{q^\omega(p, y)} \frac{\partial}{\partial y} q^\omega(p, y) \right] = y E \left[ \frac{\partial}{\partial y} \log(q^\omega(p, y)) \right] = y \frac{\partial}{\partial y} E \left[ \log(q^\omega(p, y)) \right], \]

which is identified from cross-sectional data. \(^{22}\)

Knowledge of average income effects is useful within the context of the sufficient statistic approach, where income effects enter the first-order approximations when the price or tax schedule is nonlinear. \(^{23}\) In almost all of the literature, however, income effects

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\(^{21}\)Specifications with convex compensated demands include linear and CES demand systems.

\(^{22}\)This result is related to the work of Paluch, Kneip, and Hildenbrand (2012), who derive a connection between individual and aggregate income elasticities.

\(^{23}\)For example, see Kleven (2020) for a comprehensive discussion.
are ignored by assuming that individuals have quasi-linear utilities. Expression (10) provides a means to test this assumption nonparametrically.

4.2 Non Linearity in Price (But Not Income)

The second-order approximation in the previous section works well if price changes are small or if demand is approximately linear in prices and income. In effect, the above approach only uses demand at one budget set and extrapolates linearly. However, we can compute more accurate welfare changes to accommodate large price changes.

To carry this out, we use the method introduced by Hausman (1981) and Vartia (1983). They demonstrated that the CV could also be expressed as the solution to a first-order nonlinear ordinary differential equation (ODE).

Let $p(t) : [0, 1] \rightarrow \mathcal{P}$ be a continuous price path with $p(0) = p_0$ and $p(1) = p_1$. Further, define

$$s^\omega(t) = e^\omega(p(t), v^\omega_0) - y, \quad t \in [0, 1]$$

where $v^\omega_0$ is the indirect utility at price $p$ and income $y$. We can now differentiate this expression with respect to $t$, fetching us:

$$\frac{\partial s^\omega(t)}{\partial t} = \frac{\partial}{\partial p} e^\omega(p(t), v^\omega_0) \frac{\partial p(t)}{\partial t}, \quad t \in [0, 1].$$

By Shephard’s lemma (6), the right hand side reduces to $q^\omega(p(t), y + s^\omega(t)) \frac{\partial p(t)}{\partial t}$, allowing us to write

$$\frac{\partial s^\omega(t)}{\partial t} = q^\omega(p(t), y + s^\omega(t)) \frac{\partial p(t)}{\partial t}, \quad t \in [0, 1],$$

with boundary condition $s^\omega(0) = 0$.

The CV solves this equation for $t = 1$. If an individual’s demand function is known, the change in welfare can be therefore calculated exactly.

In this reformulation, exploiting knowledge of income effects at prices along the path of the price change and not just at the original price can help improve our welfare estimates.

**Theorem 3.** Consider individual-specific income effects that are constant in prices and income: i.e., $\frac{\partial}{\partial y} q^\omega(p, y) = a^\omega_1$ for all $\omega \in \Omega$ and $p, y \in \mathcal{P} \times \mathcal{Y}$. The average CV is identified up to second order,

$$\mathbb{E}[CV^\omega(p_0, p_1, y)] = \Delta p \int_0^1 M_1(p(t), y) dt + \frac{(\Delta p)^2}{2} \int_0^1 \frac{\partial}{\partial y} M_2(p(t), y) (1 - t) dt + O((\Delta p)^3),$$

Gruber and Saez (2002) conduct a parametric test and find evidence for economically insignificant income effects. Most of the subsequent literature has therefore ignored income effects altogether (e.g., see Burns and Ziliak (2016) and the references therein).

The solution to this ODE exists an is unique when individual demand $q^\omega$ is Lipschitz in $t$ and $s$.~

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24 Gruber and Saez (2002) conduct a parametric test and find evidence for economically insignificant income effects. Most of the subsequent literature has therefore ignored income effects altogether (e.g., see Burns and Ziliak (2016) and the references therein).

25 The solution to this ODE exists an is unique when individual demand $q^\omega$ is Lipschitz in $t$ and $s$.~
where \( p(t) = p_0 + t \Delta p \) is the linear price path.

Proof. Since the income effect is assumed to be constant in prices and income, we can write \( q^\omega(p(t), y + s(t)) = q^\omega(p(t), y) + a_1^\omega s(t) \). Assuming a linear price path, Expression (12) therefore simplifies to the linear first-order ODE

\[
\frac{\partial s^\omega(t)}{\partial t} = [q^\omega(p(t), y) + a_1^\omega s(t)] \Delta p,
\]

which has the explicit solution

\[
s^\omega(t) = \exp(a_1^\omega \Delta p t) \int_0^t \Delta p \exp(-a_1^\omega \Delta p \tau) q^\omega(p(\tau), y) \, d\tau
\]

such that

\[
CV^\omega(p_0, p_1, y) = s^\omega(1) = \exp(a_1^\omega \Delta p) \int_0^1 \Delta p \exp(-a_1^\omega \Delta p t) q^\omega(p(t), y) \, dt.
\]

Given that \( \exp(x) = 1 + x + O(x^2) \), we have that

\[
CV^\omega(p_0, p_1, y) = \Delta p \int_0^1 \exp(a_1^\omega \Delta p t) q^\omega(p(t), y) \, dt
\]

\[
= \Delta p \int_0^1 [1 + a_1^\omega \Delta p(1 - t)] q^\omega(p(t), y) \, dt + O((\Delta p)^3)
\]

\[
= \Delta p \int_0^1 q^\omega(p(t), y) \, dt + (\Delta p)^2 \int_0^1 q^\omega(p(t), y) a_1^\omega (1 - t) \, dt + O((\Delta p)^3)
\]

\[
= \Delta p \int_0^1 q^\omega(p(t), y) \, dt + \frac{(\Delta p)^2}{2} \int_0^1 \frac{\partial}{\partial y} (q^\omega(p(t), y))^2 (1 - t) \, dt + O((\Delta p)^3).
\]

(14)

Taking expectations on both sides leads to the expression

\[
E[CV^\omega(p_0, p_1, y)] = \Delta p \int_0^1 M_1(p(t), y) \, dt + \frac{(\Delta p)^2}{2} \int_0^1 \frac{\partial}{\partial y} M_2(p(t), y)(1 - t) \, dt + O((\Delta p)^3).
\]

\[
\square
\]

Remark 6. Under the assumptions of Theorem 3, our approximation acts as a lower bound for the average CV. This can be readily seen from the fact that \( \exp(x) \geq 1 + x \); the second equality in Expression (14) can therefore be replaced by an inequality.

Moreover, our estimate is always below the upper bound as derived by Hausman and Newey (2016), as
\[ CV_{B_u}^ω = \Delta p \int_0^1 \exp(B_u \Delta p(1 - t)) q^ω(p(t), y) dt \]
\[ \geq \Delta p \int_0^1 [1 + B_u \Delta p(1 - t)] B_u \Delta p(1 - t) q^ω(p(t), y) dt \]
\[ \geq \Delta p \int_0^1 [1 + B_u \Delta p(1 - t)] a^ω \Delta p(1 - t) q^ω(p(t), y) dt. \]

**Remark 7.** Unfortunately, higher moments of the CV cannot be approximated using a similar approach to Theorem 3. To see this, consider the second moment of the CV. Raising both sides of Expression (13) to the second power yields
\[ (CV^ω(p_0, p_1, y))^2 = (\Delta p)^2 \int_0^1 \int_0^1 \exp(a^ω_1 \Delta p(2 - t - t')) q^ω(p(t), y)q^ω(p(t'), y) dt dt'. \]

Even the zeroth-order expansion of the exponential functions gives, after taking expectations, a term that contains \( E[q^ω(p(t), y)q^ω(p(t'), y)] \). The covariance of individual demand at different prices is not identified from cross-sectional data unless demand is assumed to be a linear function of prices.

**Remark 8.** If one does away with the assumption that the income effect is price independent, one also gets nonidentification of the second-order approximation. Let \( q^ω(p(t), y + s(t)) = q^ω(p(t), y) + a^ω_1(t)s(t) \). Analogous arguments as in the proof of Theorem 3 give
\[ CV^ω(p_0, p_1, y) = \Delta p \int_0^1 \exp(\Delta p \left( \int_0^1 a^ω_1(t') dt' - \int_0^t a^ω_1(t'') dt'' \right)) q(p(t), y) dt. \]

The term \( E \left[ \int_0^1 \int_0^1 a^ω_1(\tau_1)a^ω_1(\tau_2)q^ω(p(\tau_3), y) d\tau_1 d\tau_2 d\tau_3 \right] \) is not identified from cross-sections.

The above two remarks loosely make the point that the approximation in 3 is "tight". One cannot allow for non-linearity in income or use similar techniques to construct approximations of higher-order moments from purely cross-sectional data.

**Remark 9.** Information on average income effects can also be exploited to tighten the identified set provided by Hausman and Newey (2016). This set is derived by means of uniform bounds on individuals’ income effects. Using Chebyshev inequalities, one can restrict the probability of extreme income effects from knowledge of these bounds and the observed average income effect. This, in turn, restricts the probability of extreme values for the CV. The resulting set is probabilistic in the sense that it comes along with a coverage probability for the true average CV to be within the set.
Formally, let $B_\omega(t, s) = \Delta p \partial q(p(t), y + s)$ and let $B_u^\omega = \sup_{t, s} B_\omega(t, s)$ and $B_l^\omega = \inf_{t, s} B_\omega(t, s)$. Assuming income effects to be contained within $[B, \overline{B}]$ with $\overline{B} \geq 0$, and using Chebyshev’s inequality for bounded variables, we have that

$$\Pr[B_u^\omega \geq k] \geq \frac{\mathbb{E}[B_u^\omega] - k}{\overline{B}} \geq \frac{\sup_{t, s} \mathbb{E}[B^\omega(t, s)] - k}{\overline{B}},$$

and

$$\Pr[B_l^\omega \geq z] \leq \frac{\mathbb{E}[B_l^\omega]}{z} \leq \frac{\inf_{t, s} \mathbb{E}[B^\omega(t, s)]}{z},$$

where both right-hand sides are identified from cross-sectional data. From Theorem 3 in Hausman and Newey (2016), we know that $CV_{B_l}(p_0, p_1, y) \leq CV_{B_u}(p_0, p_1, y) \leq CV_{B_u}(p_0, p_1, y)$ for $\Delta p > 0$, such that

$$\mathbb{E}[CV^\omega] \geq \Pr[B_u^\omega \geq k] \mathbb{E}[CV^\omega_{B_u}] \geq \Pr[B_u^\omega \geq k] \mathbb{E}[CV^\omega_{B_u}] \geq \Pr[B_u^\omega \geq k] \mathbb{E}[CV^\omega_{B_u}] \geq \Pr[B_u^\omega \geq k] \mathbb{E}[CV^\omega_{B_u} | B_u^\omega = \overline{B}],$$

and

$$\mathbb{E}[CV^\omega] \leq \mathbb{E}[CV^\omega_{B_u}] \leq \Pr[B_u^\omega \geq k] \mathbb{E}[CV^\omega_{B_u} | B_u^\omega = \overline{B}] + (1 - \Pr[B_u^\omega \geq k]) \mathbb{E}[CV^\omega_{B_u} | B_u^\omega \geq \overline{B}],$$

where the dependence of the CV on prices and income is suppressed for notational clarity. By varying $z$ and $k$, these bounds can be computed for arbitrary degrees of statistical coverage. Note that by setting $z = \overline{B}$ and $k = \overline{B}$, one obtains the bounds of Hausman and Newey (2016) as a special case.

### 5 Conditional Moments and Rationality

In this section, we study how the conditional moments of demand can be used to test the rationality of a population. Hurwicz and Uzawa (1971) provide well-known necessary and sufficient conditions for the integrability of demand.

In the case where the analyst can observe conditional quantile demand functions, this problem has been studied by Dette, Hoderlein, and Neumeyer (2016) and Hausman and Newey (2016). We contribute to the literature by considering the empirical content of moments instead of quantiles.

#### 5.1 Two-good Case

Assuming homogeneity of degree zero and Walras’ law hold, the only remaining restriction is negative semidefiniteness, as symmetry holds trivially in the two-good case. In
particular, for all types \( \omega \in \Omega \), and for all budget sets \((p, y) \in \mathcal{P} \times \mathcal{Y}\),
\[
\frac{\partial}{\partial p} q^\omega(p, y) + q^\omega(p, y) \frac{\partial}{\partial y} q^\omega(p, y) \leq 0.
\]
This restriction can be rewritten in terms of the conditional moments of demand. Multiplying both sides by \( q^\omega(p, y) \) for some \( n \in \mathbb{N}_+ \), we define
\[
\Gamma_n^\omega(p, y) = q^\omega(p, y)^n \left[ \frac{\partial}{\partial p} q^\omega(p, y) + q^\omega(p, y) \frac{\partial}{\partial y} q^\omega(p, y) \right],
\]
and
\[
\Gamma_n(p, y) = \mathbb{E} [\Gamma_n^\omega(p, y)].
\]
Since for every type,
\[
\Gamma_n^\omega(p, y) = \frac{1}{n+1} \frac{\partial}{\partial p} q(p, y)^{n+1} + \frac{1}{n+2} \frac{\partial}{\partial y} q(p, y)^{n+2} \leq 0,
\]
we have that
\[
\Gamma_n(p, y) = \int \left( \frac{1}{n+1} \frac{\partial}{\partial p} M_{n+1}(p, y) + \frac{1}{n+2} \frac{\partial}{\partial y} M_{n+2}(p, y) \right) dF(\omega)
= \frac{1}{n+1} \frac{\partial}{\partial p} M_{n+1}(p, y) + \frac{1}{n+2} \frac{\partial}{\partial y} M_{n+2}(p, y)
\leq 0,
\]
where the second equality follows from interchanging integration and differentiation as well as the definition of the conditional moments, and the inequality follows from the Slutsky equation being point-wise negative. This expression imposes a necessary restriction on every two consecutive moments. Notice that \( \Gamma_n(p, y) \) maps a budget set to a real number.

More generally, let \( \mathbb{Q}[\mathbb{R}] \) be the set of polynomials over the real numbers with rational coefficients that are positive in the support of demand \([0, \frac{y}{p}]\). For any polynomial \( \pi_n^\omega(p, y) = \sum_{i=1}^n a_i(q^\omega(p, y))^n \in \mathbb{Q}[\mathbb{R}] \), we define
\[
\Lambda_n^\omega(p, y) = \pi_n^\omega(p, y) \left[ \frac{\partial}{\partial p} q^\omega(p, y) + q^\omega(p, y) \frac{\partial}{\partial y} q^\omega(p, y) \right],
\]
and
\[
\Lambda_n(p, y) = \mathbb{E} [\Lambda_n^\omega(p, y)].
\]
We can use the linearity of the expectation to compute $\Lambda_{\pi_n}(p, y)$ and $\Lambda_{\pi_n}(p, y)$.

$$\pi_n(p, y) = \sum_{i=1}^{n} a_i(q^\omega(p, y))^n$$

$$\implies \Lambda_{\pi_n}(p, y) = \sum_{i=1}^{n} a_i \Gamma_n(p, y)$$

and

$$\Lambda_{\pi_n}(p, y) = \mathbb{E}[\Lambda_{\pi_n}(p, y)] = \sum_{i=1}^{n} a_i \Gamma_n(p, y)$$

This allows us to characterize demand in terms of moments of demand.

**Theorem 4.** In the two-good case, the following statements are equivalent:

1. A demand distribution can be generated by a rational population.

2. For any polynomial $\pi_n(p, y)$ that is positive in the support of the distribution of demand at $(p, y)$, it holds that $\Lambda_{\pi_n}(p, y) \leq 0$.

**Proof.** The $(1) \implies (2)$ part simply follows from any polynomial transformation being a sum of monomial transformations, thus requiring negativity. For the $(2) \implies (1)$ part, we proceed by means of proof by contradiction. Hausman and Newey (2016) show that negativity of the quantile demand function characterizes rationalizability. Suppose $(2)$ holds, but negativity is contradicted at some quantile. This would mean that there is some quantile $\tau \in (0, 1)$, and some quantile demand $\tilde{q}(\tau \mid p, y) = \inf \{q : \Pr[q^{\omega}(p, y) \leq q \mid p, y] \geq \tau\}$ such that

$$\frac{\partial}{\partial p} \tilde{q}(\tau \mid p, y) + \tilde{q}(\tau \mid p, y) \frac{\partial}{\partial y} \tilde{q}(\tau \mid p, y) > 0.$$  

We can pick a sequence of polynomials $\{\pi_n\}_{n=1}^{\infty}$ such that\footnote{To be precise, one should pick a set of sequences of polynomials that uniformly converge in a neighborhood of $p, y$. Therefore, derivatives with respect to elements of $p, y$ are well-defined.}

$$\lim_{n \to \infty} \{\pi_n\} \to \delta(\tilde{q}(\tau | p, y)),$$

where $\delta$ is the Dirac delta function. Therefore, by continuity of $\Lambda_{\pi_n}$, we have that

$$\lim_{n \to \infty} \{\Lambda_{\pi_n}(p, y)\} \to \left[ \frac{\partial}{\partial p} \tilde{q}(\tau \mid p, y) + \tilde{q}(\tau \mid p, y) \frac{\partial}{\partial y} \tilde{q}(\tau \mid p, y) \right] > 0,$$

which means that beyond some finite $n \in \mathbb{N}_+$, negativity must be contradicted. This would in turn contradict $(2)$, hence proving the theorem. \qed
Remark 10. The equivalence in Theorem 4 can be used to construct a semi-decidable test.\footnote{This test has the property that no rationalizable distribution is ever rejected, and all non-rationalizable distributions are eventually rejected.} Let $Q_+ [\mathbb{R}] = \{ \pi \in Q [\mathbb{R}] \mid x \in [0, y/p] \implies \pi(x) \geq 0 \}$ be the set of polynomials over the real numbers with rational coefficients that are positive for $x \geq 0$. Since the rational numbers are countable, so is the set $Q_+ [\mathbb{R}]$; one can therefore pick an enumeration $\{ \pi_n \}_{n=1}^{\infty}$ of this set. A simple semi-decidable test would consist of the following iterative scheme at step $n$:

1. If $\Lambda(\pi_n, p, y) \leq 0$, move to the $(n + 1)$st step;
2. If $\Lambda(\pi_n, p, y) > 0$, stop and reject the distribution.

The first part follows directly from Theorem 4. The second part follows from the fact that if the distribution is not rationalizable, there exists some polynomial $\pi$ which has a positive translation. Since $\{ \pi_n \}_{n=1}^{\infty}$ is countable, there must exist some $n$ where $\pi_n$ has a positive translation, leading to rejection.

Remark 11. In the case where only the zeroth and first monomial translation can be computed (or equivalently, the first three moments can be observed), only linear polynomials enter the analysis, which makes testing much simpler. Denote the support of demand at budget set $p, y$ as $0 \leq q_{\min} \leq q_{\max} \leq y/p$. In terms of the first two translations, only four polynomials need to be checked for negativity: (i) $1$; (ii) $x$; (iii) $-q_{\min} + x$; and (iv) $q_{\max} - x$. This translates to the conditions:

\[
\Lambda(1, p, y) \leq 0, \\
\Lambda(x, p, y) \leq 0, \\
-q_{\min} \Lambda(1, p, y) + \Lambda(x, p, y) \leq 0, \\
q_{\max} \Lambda(1, p, y) - \Lambda(x, p, y) \leq 0.
\]

This means that in addition to monomial negativity, only $q_{\max} \Lambda(1, p, y) \leq \Lambda(x, p, y) \leq q_{\min} \Lambda(1, p, y)$ needs to be checked. Figure 2 shows the admissible set of solutions shaded in red.

5.2 Many-good Case

We now consider the case where we have multiple goods. From the Slutsky equation (5), we have that

\[
\mathbb{E} \left[ \frac{\partial}{\partial p} h^\omega(p, u) \right] = \mathbb{E} \left[ \frac{\partial}{\partial p} q^\omega(p, y) \right] + \mathbb{E} \left[ \frac{\partial}{\partial y} q^\omega(p, y)(q^\omega(p, y))^T \right].
\]
Without imposing Slutsky symmetry, \( E \left[ \frac{\partial}{\partial p} q^\omega(p, y) \right] \) is not automatically identified from the first two conditional moments of demand. This is due to the fact that the variance of demand \( M_2 \) being symmetric imposes a loss of “degrees of freedom”. This is different from the two-good case, where there is no loss of information because symmetry holds trivially.

**Proposition 1.** Without Slutsky symmetry being imposed,

\[
E \left[ \frac{\partial}{\partial p} q^\omega(p, y) + \frac{\partial}{\partial y} q^\omega(p, y)(q^\omega(p, y))^\top \right]
\]

is not identified from the first two moments of demand.

**Proof.** For simplicity, we consider the two-good case. Firstly, observe that the first part of the expectation, namely \( E \left[ \frac{\partial}{\partial p} q^\omega(p, y) \right] \) is identified because it is simply the price derivative of the first moment of demand.

From the definition of the conditional variance,

\[
M_2(p, y) = E \begin{bmatrix} (q_1^\omega(p, y))^2 & q_1^\omega(p, y)q_2^\omega(p, y) \\ q_1^\omega(p, y)q_2^\omega(p, y) & (q_2^\omega(p, y))^2 \end{bmatrix},
\]
which is a symmetric matrix. However, one needs to identify,
\[
\mathbb{E} \begin{bmatrix}
q_1^\omega(p, y) \frac{\partial}{\partial y} q_1^\omega(p, y) & q_1^\omega(p, y) \frac{\partial}{\partial y} q_2^\omega(p, y) \\
q_2^\omega(p, y) \frac{\partial}{\partial y} q_1^\omega(p, y) & q_2^\omega(p, y) \frac{\partial}{\partial y} q_2^\omega(p, y)
\end{bmatrix}
\]

Even though the diagonal terms of the matrix are pinned down, the off-diagonal terms cannot be identified because the information in the variance is redundant. In particular, we can identify
\[
\mathbb{E} \left[ \frac{\partial}{\partial y} q_1^\omega(p, y) q_2^\omega(p, y) \right] = \mathbb{E} \left[ q_1^\omega(p, y) \frac{\partial}{\partial y} q_2^\omega(p, y) + \frac{\partial}{\partial y} q_1^\omega(p, y) q_2^\omega(p, y) \right],
\]
but not the terms at the right-hand side separately. This means there can exist different models that disagree on the value of \( \frac{\partial}{\partial y} q_1^\omega(p, y) q_2^\omega(p, y)^\top \) but are observationally equivalent in terms of mean and variance.

This result shows that if we remain agnostic about rationality, the above term is not identified from the first two moments of demand. However, if we assume that individuals satisfy Slutsky symmetry, this exactly identifies the Slutsky terms. This leads to the following theorem (which already appeared before as Lemma 3).

**Theorem 5.** If individuals obey Slutsky symmetry, the first two moments identify the Slutsky matrix \( \mathbb{E} \left[ \frac{\partial}{\partial p} h^\omega(p, u) \right] \).

**Proof.** Using the definition of the conditional moments in Expression (7), we know that
\[
\frac{\partial}{\partial y} M_2(p, y) = \frac{\partial}{\partial y} \left( \int q_\omega^\omega(p, y)(q_\omega^\omega(p, y))^\top dF(\omega) \right).
\]
Interchanging the derivative and integral operators gives us
\[
\frac{\partial}{\partial y} M_2(p, y) = \int \frac{\partial}{\partial y} (q_\omega^\omega(p, y)(q_\omega^\omega(p, y))^\top) dF(\omega)
= \int \omega \left[ \frac{\partial}{\partial y} q_\omega^\omega(p, y)(q_\omega^\omega(p, y))^\top + q_\omega^\omega(p, y) \left( \frac{\partial}{\partial y} q_\omega^\omega(p, y) \right)^\top \right] dF(\omega).
\]
From the Slutsky equation (5), we have that
\[
\frac{\partial}{\partial p} h^\omega(p, u) = \frac{\partial}{\partial p} q_\omega^\omega(p, y) + \frac{\partial}{\partial y} q_\omega^\omega(p, y)(q_\omega^\omega(p, y))^\top,
\]
which is symmetric due to Slutsky symmetry. Adding this equation to its transpose fetches us
\[
2 \frac{\partial}{\partial p} h^\omega(p, u) = \frac{\partial}{\partial p} q_\omega^\omega(p, y) + \left( \frac{\partial}{\partial p} q_\omega^\omega(p, y) \right)^\top + \frac{\partial}{\partial y} q_\omega^\omega(p, y)(q_\omega^\omega(p, y))^\top + q_\omega^\omega(p, y) \left( \frac{\partial}{\partial y} q_\omega^\omega(p, y) \right)^\top,
\]
\]
such that

\[
\mathbb{E} \left[ \frac{\partial}{\partial p} h^\omega(p,u) \right] = \frac{1}{2} \left[ \frac{\partial}{\partial p} M_1(p,y) + (\frac{\partial}{\partial p} M_1(p,y))^\top + \frac{\partial}{\partial y} M_2(p,y) \right].
\]

\[\square\]

Remark 12. Proposition 1, together with the above theorem implies that Slutsky symmetry is untestable from the first two conditional moments of demand. That can be seen because there are several values of \( \mathbb{E}[q^\omega \frac{\partial}{\partial y} q^\omega] \) which agree with a mean-variance system, but only one value of \( \mathbb{E}[q^\omega \frac{\partial}{\partial y} q^\omega] \) arises from a symmetric system. Therefore there must be asymmetric systems which agree with the mean-variance data, and there must also be symmetric systems as we have shown above. This renders symmetry untestable.

Remark 13. Theorem 5 shows that two symmetric models that generate the same conditional mean and variance of demand (e.g., see the example in Lemma 4) have the same average substitution. Therefore, the first two moments pin down average substitution under Slutsky symmetry. Figure 3 provides a Venn diagram of these results.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{venn_diagram.png}
\caption{Symmetry and rationalizability}
\end{figure}

Remark 14. Assuming Slutsky symmetry, one can test negative semi-definiteness of the population based on the first two moments of demand. If they are rationalizable,

\[
P(p, y) = \frac{\partial}{\partial p} M_1(p, y) + \frac{1}{2} \frac{\partial}{\partial y} M_2(p, y)
\]
must be negative semidefinite. This follows from the fact that \( \frac{\partial}{\partial p} \omega(p, u) \) is negative semidefinite and \( P(p, y) + P(p, y)^T = 2 \frac{\partial}{\partial p} \omega(p, u) \).

Akin to the two-goods case, we have similar restrictions on the higher moments of demand. The difference is that the monomial translation for any moment is now a tensor form. The following theorem provides necessary conditions for the moments to be generated by a demand system.

**Theorem 6.** For any \( n \in \mathbb{N}_+ \), the following \( n + 1 \) tensor form is negative semidefinite.\(^{29}\)

\[
n^{-1} \frac{\partial}{\partial p} M_n + (n + 1)^{-1} \frac{\partial}{\partial y} M_{n+1}
\]

**Proof.** The proof is relegated to Appendix C. \( \square \)

Notice that the form is \( n + 1 \) because differentiating a \( n \) form with respect to price increases the order of the form.

**Remark 15.** Because the restriction in Theorem 6 is a test of negative semidefiniteness (and not of symmetry), any small perturbation of a finite and rationalizable moment sequence is itself also rationalizable. This is because negative semidefiniteness is an open condition.

**Remark 16.** Finally, the restrictions in Theorems 4 and 6 do not depend on the levels of the moments, but only on their changes with respect to prices and income. This leads to two fundamental properties of these restrictions. First, if there is additively separable i.i.d. measurement error in the observed demands, these restrictions can still be estimated consistently. Second, none of our restrictions depend on statistical constraints on moments, such as non-negativity (for even moments) or Chebyshev-type tail inequalities.

### 6 Empirical Illustration

We now apply our results on consumer data from cross-sectional household budget surveys. In Section 6.1, we first describe our data and lay out the estimation procedure. Section 6.2 outlines the price changes and compares our results with those of the RA approach.

\(^{28}\)Note that for a square matrix \( A \) it holds that \( v^T (A + A^T) v = 2v^T Av \).

\(^{29}\)We say a tensor form \( T_n \) is negative semidefinite if

\[
T_n(v \times v \times \cdots \times v) = \sum_{i_1, i_2, \ldots, i_n = 1}^{l-1} t_{i_1, i_2, \ldots, i_n} v_{i_1} v_{i_2} \cdots v_{i_n} \leq 0, \quad \forall v \in \mathbb{R}^{l-1}.
\]
6.1 Data and Estimation

The data we employ consists of repeated cross-sections of household budget surveys from the UK. In particular, we use 14 waves from the Expenditure and Food Survey (2006-2007) and the Living Costs and Food Survey (2008-2019). These contain detailed observations on households’ expenditures, income, and demographic characteristics. Price data is collected from the Office for National Statistics.

We aggregate households’ expenditures into four broad categories: (i) food, (ii) housing, (iii) transportation, and (iv) other nondurables and services. “Food” consists of expenditures on food and non-alcoholic beverages. “Housing” encompasses goods and services for the usage, maintenance, supply of water, and heating of the household’s dwelling. “Transport” covers the purchase of vehicles and expenditures on maintenance and fuel, passenger transport services, and courier services. “Other nondurables and services” encompasses expenditures on alcohol, tobacco, clothing and footwear, communication, recreation, and restaurants and hotels.\(^{30}\)

To facilitate nonparametric estimation, we impose some sample restrictions. We drop households with zero expenditures on rent or transportation. We also remove those households with budget shares outside the 2nd-98th percentile range for at least one of the four categories. To further reduce the influence of outliers, we trim households with total expenditures and disposable income outside the 2nd-98th percentile range. Our final estimation sample consists of 12,494 households; descriptive statistics are relegated to Appendix E.

To apply our method for average welfare, we need to estimate the first two conditional moments of demand. The conditional means and variances are modelled semiparametrically using partially linear kernel regression (Racine and Li, 2004).\(^{31}\) This specification is flexible in budget sets and avoids the curse of dimensionality. In particular, for every category \(k\) we have

\[
\mathbb{E}[q_k^\omega(p, y, w)^{n}] = h_k(p, y) + \delta_k^n d, \quad n = 1, 2, \quad (16)
\]

where \(h_k\) is a nonparametric function in prices and income, and \(\delta_k\) is a vector that captures the impact of household characteristics \(d\).\(^{32}\) The latter controls for observed heterogeneity and consists of the number of adults, number of children, number of retired, and number of earners in the household.

We calculate the required price and income derivatives on the basis of the estimated moment functions in Expression (16). Following Paluch, Kneip, and Hildenbrand (2012), we test whether our results are sensitive to outlying values of these derivatives. We find

\(^{30}\)More details on the construction of these categories is provided in Appendix E.

\(^{31}\)As a consequence of Remark 1, it suffices to model the mean and variance for every category separately if only a single price is changed at a time.

\(^{32}\)The category “other nondurables and services” will be treated as the numeraire.
that trimming derivatives outside the 5th-95th percentile range does not qualitatively change our results.

A disadvantage of our price data is that it contains does not contain household-level variation. To increase cross-sectional price variation, we make use of Stone-Lewbel price indices (Hoderlein and Mihaleva, 2008). These household-specific indices make use of the variability in expenditures on nested commodities. For every category we will use the variation in budget shares of commodities one COICOP level lower.\textsuperscript{33} To ensure tractability, we will assume that the within-category preferences over these nested commodities are Cobb-Douglas.\textsuperscript{34} Notice that in this approach, the between-category preferences remain arbitrarily flexible.

6.2 Empirical Results

We focus on the welfare impact of six distinct scenarios: a 5 or 10% increase in the price of either food, housing, or transportation. In every scenario the price of the other goods is kept constant. To allow for meaningful interpersonal welfare comparisons, for every household, we fix the vector of baseline prices to the sample mean.

Table 2 shows the estimates for the average CV using the RA and our approach. As the demands for all goods are normal, we find that the estimate for the RA approach is below our estimate in each of the six scenarios. This is especially true for housing and transportation where the relative bias can be as high as 17.9\% (5\% increase for transportation) or 27.2\% (10\% increase for transportation). These differences imply that the RA approach might significantly understate the welfare cost of the price increases.

Moreover, as depicted in Figure 5, we find that this relative bias has a distributional dimension. For food, we find that the bias is larger for households with substantial weekly incomes. For transportation the bias is highest for those household with a weekly disposable income of around 300 pounds.

In the Appendix, we provide more insight in these distributional patterns by plotting the variance in consumption with respect to income. It is shown that the error between both approaches is proportional to these variance under common parametric specifications (e.g., the Almost Ideal Demand System).

\textsuperscript{33}The Classification of Individual Consumption According to Purpose (COICOP) harmonizes the classification of household expenditures across countries.

\textsuperscript{34}The assumption of Cobb-Douglas preferences delivers a simple expression for the price indices. Denote \( q_{ikj} \) the demand of individual \( i \) for good \( j \) in category \( k \) and let \( p_{kj} \) be the price of this good. The price index for the \( i \)th individual for category \( k \) becomes

\[
p_{ik} = \left( \prod_{j=1}^{n_k} \frac{w_{kj}}{\overline{w}_{kj}} \right)^{-1} \prod_{j=1}^{n_k} \left( \frac{p_{kj}}{w_{ikj}} \right)^{w_{ikj}},
\]

where \( w_{ikj} = \frac{q_{ikj}p_{kj}}{\sum_{j=1}^{n_k} q_{ikj}p_{kj}} \) and \( \overline{w}_{kj} = \frac{1}{n} \sum_{i=1}^{n} w_{ikj} \).
Figure 4: Relative deviation in average welfare by income level (5% price increases)
Table 2: Average welfare impact of a 5 and 10% price increase

<table>
<thead>
<tr>
<th>category</th>
<th>E[CV]</th>
<th>RA approach</th>
<th>our approach</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>5% price increase</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>food</td>
<td>3.19</td>
<td>3.22</td>
<td></td>
</tr>
<tr>
<td>housing</td>
<td>11.31</td>
<td>12.38</td>
<td></td>
</tr>
<tr>
<td>transportation</td>
<td>4.31</td>
<td>5.08</td>
<td></td>
</tr>
<tr>
<td><strong>10% price increase</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>food</td>
<td>7.06</td>
<td>7.20</td>
<td></td>
</tr>
<tr>
<td>housing</td>
<td>30.75</td>
<td>35.00</td>
<td></td>
</tr>
<tr>
<td>transportation</td>
<td>10.66</td>
<td>13.57</td>
<td></td>
</tr>
</tbody>
</table>

7 Conclusion

In this paper, we introduce novel methods to approximate welfare changes which are caused by price changes. To do so, we show that the conditional moments of demand contain information about the distribution of individuals’ income effects. We use this information to conduct more accurate counterfactual exercises in applied welfare analysis. We also demonstrate that better approximations cannot be found using cross-sections. Furthermore, we show that the conditional moments of demand contain empirical content and can be used to test individual rationality, specifically the negative semidefiniteness of the Slutsky matrix.

Going forward, there is room for future work in at least two directions. Firstly, it would be interesting to understand what additional power short panels on consumers would give to estimate the above counterfactuals; first steps in this direction has been made by Crawford (2019) and Cooprider, Hoderlein, and Alexander (2022). Considering stochastic rationalizability, it is still a wide-open question as to whether Slutsky symmetry can be tested on cross-sectional data and if so, how to construct tests. Asking if symmetry carries any empirical content at the level of cross-sections would in itself be a very interesting question.
References


Appendix

*Price Changes and Welfare Analysis: Measurement under Individual Heterogeneity*

A Regularity Conditions

Every individual’s demand function \( q^\omega \) needs to be infinitely differentiable in \( p, y \) at all \( p, y \in P \times Y \). This is ensured by the following condition.

**Assumption A.1.** Every individual’s preferences are continuous, strictly convex, and locally nonsatiated. The associated utility functions \( u^\omega \) are infinitely differentiable everywhere.

The following condition ensures that the dominated convergence theorem holds. This allows us to interchange limits and integrals.

**Assumption A.2.** There exists a function \( g : \Omega \to \mathbb{R} \) such that for all \( p, y \in P \times Y \) and \( n, m \in \mathbb{N} \) it holds that

\[
\| \text{vec}(D_{p^n,y^m}q^\omega(p,y)) \| \leq g(\omega) \quad \text{with} \quad \int g(\omega)dF(\omega) < \infty.
\]

Finally, we require that all moments exist and are finite.

**Assumption A.3.** For all \( n \in \mathbb{N} \), it holds that

\[
\mathbb{E}[|T_n^\omega(p,y)|] < \infty.
\]

B Results for Welfare in the Many-goods Case

We now analyze the distribution of the compensating variation in the case where there are more than two goods. This requires what we refer to as a *symmetrization procedure*: i.e., to obtain an estimate of the average substitution effect, we need to impose Slutsky symmetry.\(^{35}\)

**Lemma 3.** Analogously with the two-goods case, the following holds for three or more

\(^{35}\)See Section 5 for a more detailed discussion on the role of Slutsky symmetry.
goods,
\[
E \left[ \frac{\partial}{\partial p} h^\omega(p, u) \right] = \frac{1}{2} \left( \frac{\partial}{\partial p} M_1(p, y) + \frac{\partial}{\partial p} M_1(p, y)^\top + \frac{\partial}{\partial y} M_2(p, y) \right).
\]

**Proof.** Refer to the proof of Theorem 5 in Section 5. \qed

A similar symmetrization procedure is needed to obtain second-order approximations for all moments of the compensating variation. This is provided in Theorem 6 in Section 5.

**Corollary 6.1.** In the many-good case, the second-order approximation of the \(n\)th moment of the compensating variation depends *only* on the \(n\)th and \((n+1)\)th conditional moment of demand. Formally, we have that

\[
E[CV_\omega(p_0, p_1, y)^n] = p^{(**)} \left\{ \frac{1}{n} \left[ \sum (\text{symmetrized moment derivative}) \right] + \frac{n}{n+1} \frac{\partial}{\partial y} M_{n+1} \right\}.
\]

**Proof.** The proof is similar to that of Theorem 1 and is relegated to Section C in the Appendix. \qed

**Remark 17.** When the prices of all goods change, the second-order approximation for the average compensating variation requires estimating the entire variance-covariance matrix, which might be burdensome. However, it is possible to bound the off-diagonal elements of this matrix from the marginal conditional variances. In particular, one can impose the following restrictions:

\[
\begin{align*}
[M^2(p, y)]_{ij} &= [M^2(p, y)]_{ji}, \\
\mathbf{p} \cdot \frac{\partial}{\partial y} M_2(p, y) &= \frac{\partial}{\partial y} M_2(p, y) \cdot \mathbf{p} = 0, \\
[M^2(p, y)]_{ij} &\leq \sqrt{[M^2(p, y)_{ii} M^2(p, y)]_{jj}}.
\end{align*}
\]

The first restriction follows the symmetry of the variance-covariance matrix, the second is the budget constraint, and the third is due to the Cauchy-Schwarz inequality. Note that the Cauchy-Schwarz inequality ensures that the off-diagonal elements have bounded support, even if we would only observe the diagonal elements.
C Proofs

C.1 Proof for Theorem 1

We have that the \( n \)th moment of the CV is equal to

\[
\mathbb{E}[CV^\omega(p_0, p_1, y)^n] = (\Delta p)^n\mathbb{E}\left[\left(\int_0^1 h^\omega(p_0 + t\Delta p, v_0^\omega)dt\right)^n\right]
\]

\[
= (\Delta p)^n\mathbb{E}\left[\int_0^1 \cdots \int_0^1 \prod_{i=1}^n h^\omega(p_0 + t_i\Delta p, v_0^\omega)dt_1 \cdots dt_n\right].
\]

Using a first-order Taylor approximation around \( t_i = 0 \) for all \( i \), we have that

\[
\prod_{i=1}^n h^\omega(p_0 + t_i\Delta p, v_0^\omega) = h^\omega(p_0, v_0^\omega)^n + \Delta p \left(\sum_{i=1}^n t_i\right) h^\omega(p_0, v_0^\omega)^{n-1} \frac{\partial}{\partial p} h^\omega(p_0, v_0^\omega) + O((\Delta p)^2)
\]

\[
= q^\omega(p_0, y)^n + \Delta p \left(\sum_{i=1}^n t_i\right) q^\omega(p_0, y)^{n-1} \frac{\partial}{\partial p} h^\omega(p_0, v_0^\omega) + O((\Delta p)^2).
\]

Putting things together, we have that

\[
\mathbb{E}[CV^\omega(p_0, p_1, y)^n] = (\Delta p)^n\mathbb{E}\left[\int_0^1 \cdots \int_0^1 q^\omega(p_0, y)^n + \Delta p \left(\sum_{i=1}^n t_i\right) q^\omega(p_0, y)^{n-1} \frac{\partial}{\partial p} h^\omega(p_0, v_0^\omega)
\]

\[
+ O((\Delta p)^2)dt_1 \cdots dt_n\right]
\]

\[
= (\Delta p)^n \left[M_n(p_0, y) + \frac{\Delta p}{2} \left(\frac{\partial}{\partial p} M_n(p_0, y) + \frac{n}{n+1} \frac{\partial}{\partial y} M_{n+1}(p_0, y)\right) + O((\Delta p)^2)\right],
\]

where the last equality follows from the Slutsky equation.

C.2 Proof of Corollary 6.1

With more than two goods, demand is a vector, which fetches us the formula

\[
\mathbb{E}[CV^\omega(p_0, p_1, y)^n] = \mathbb{E}\left[\left(\int_0^1 \Delta p \cdot h^\omega(p_0 + t\Delta p, v_0^\omega)dt\right)^n\right].
\]

Again, using the first-order expansion of compensated demand around \( t = 1 \), we have that

\[
\mathbb{E}[CV^\omega(p_0, p_1, y)^n] \approx [\Delta p \cdot h^\omega(p_0, v_0^\omega)]^n
\]

\[
+ \sum_{i=1}^n (\Delta p \cdot h^\omega(p_0, v_0^\omega))^i \left[\Delta p \frac{\partial}{\partial p} h^\omega(p_0, v_0^\omega)(\Delta p)^i\right] (\Delta p \cdot h^\omega(p_0, v_0^\omega))^{n-i-1}.
\]

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Plugging in the Slutsky equation in the second term gives
\[ \sum_{i=1}^{n} (\Delta p \cdot q'(p_0, y))^i \left[ \Delta p \left( \frac{\partial}{\partial p} q'(p_0, y) + \frac{\partial}{\partial y} q'(p_0, y) \right) \right] \left( \Delta p \cdot q'(p_0, y) \right)^{n-i-1}, \]
or after expanding,
\[ \sum_{i=1}^{n} (\Delta p \cdot q'(p_0, y))^i (\Delta p \left( \frac{\partial}{\partial p} q'(p_0, y) \right) \left( \Delta p \cdot q'(p_0, y) \right)^{n-i-1} \]
\[ + \sum_{i=1}^{n} (\Delta p \cdot q'(p_0, y))^i \left( \frac{\partial}{\partial y} q'(p_0, y) \right) \left( \Delta p \cdot q'(p_0, y) \right)^{n-i-1}. \]
The analyst, however, observes
\[ \frac{\partial}{\partial y} M_{n+1} = \sum_{i=1}^{n} \mathbb{E} \left[ \left( \bigotimes_{k=1}^{i} q'(p, y) \right) \bigotimes_{k=1}^{n-i-1} \frac{\partial}{\partial y} \left( \bigotimes_{k=1}^{n-i-1} q'(p, y) \right) \right]. \]
As with the variance, one can be written in terms of the other by means of symmetrization, giving us
\[ \Delta p (**) \frac{1}{n} \sum_{k=1}^{n} \frac{\partial}{\partial p} M_n(p, y)^{T_k} + \frac{n}{n+1} \frac{\partial}{\partial y} (M_{n+1})(p, y), \]
where (**) is the generalized tensor form and \( \frac{1}{n} \sum_{k=1}^{n} \frac{\partial}{\partial p} M_n(b)^{T_k} \) is the symmetrized version of the tensor.

Notice that for higher-order tensors, in order to symmetrize them, we need to carry out a cyclic transformation which sends element \( a_{i_1, i_2, \ldots, i_k} \rightarrow a_{i_{k+1}, i_1, \ldots, i_k} \). There are \( k \) such transformations, hence they sum up to \( k \).

**Proof of Theorem 9**
The proof follows directly from the intuition of the results we have for 2 goods.

Notice that
\[ n^{-1} \frac{\partial}{\partial p} M_n + (n+1)^{-1} \frac{\partial}{\partial y} M_{n+1} \]
is the same as the symmetrized sum
\[ \left( \bigotimes_{k=1}^{n-i-1} q'(p, y) \right) (**) \left( \frac{\partial}{\partial p} M_1 + (n+1)^{-1} \frac{\partial}{\partial y} M_2 \right) \]
where the second term is nsd by the slutsky equation.

Because the product of a NSD matrix and any other tensor form must be NSD, so must the above expression.
D Additional results

**Lemma 4.** Suppose Assumption 1 holds. Then \( \{E[q^\omega(p, y)(\frac{\partial}{\partial y} q^\omega(p, y))^n]) \}_{n=2}^\infty \) is not nonparametrically identified from cross-sectional data.

**Proof.** We show nonidentification of \( \{E[q^\omega(p, y)(\frac{\partial}{\partial y} q^\omega(p, y))^n]) \}_{n=2}^\infty \) by means of a counterexample. Suppose individual demand is linear in price and income

\[
q^\omega(p, y) = \omega_1 - p + \omega_2 y,
\]

and let \( \omega_1 \sim U(0, 1) \), and \( \Pr[\omega_2 = 1/3] = \Pr[\omega_2 = 2/3] = 1/2 \). Hausman and Newey (2016) show that for \( y < 3 \), an observationally equivalent specification is the quantile demand

\[
\tilde{q}^\omega(p, y) = \begin{cases} 
-p + \mathbb{I}[y < 6\tilde{\omega}](y/2 + \tilde{\omega}) + \mathbb{I}[y \geq 6\tilde{\omega}](y/3 + 2\tilde{\omega}), & \tilde{\omega} \leq 1/2, \\
-p + \mathbb{I}[y < 6(1 - \tilde{\omega})](y/2 + \tilde{\omega}) + \mathbb{I}[y \geq 6(1 - \tilde{\omega})](2y/3 + 2\tilde{\omega} - 1), & \tilde{\omega} > 1/2,
\end{cases}
\]

where \( \tilde{\omega} \sim U(0, 1) \).

For a budget set \((p, y) = (1, 2)\), elementary calculations show that

\[
E \left[ q^\omega(p, y) \left( \frac{\partial}{\partial y} q^\omega(p, y) \right)^n \right] = E[(\omega_1 - p + \omega_2 y)\omega^2 y^2 | p = 1, y = 2]
\]

\[
= (E[\omega_1] - 1)E[\omega^2] + 2E[\omega^3]
\]

\[
= -1/4[[1/3]^n + (2/3)^n] + [(1/3)^{n+1} + (2/3)^{n+1}]
\]

\[
= 1/12(1/3)^n + 5/12(2/3)^n
\]

holds for the original demand specification. However, after differentiating the quantile demand with respect to income, we obtain

\[
\tilde{q}^\omega(p, y) \left( \frac{\partial}{\partial y} \tilde{q}^\omega(p, y) \right)^n |_{p=1, y=2} = \begin{cases} 
\mathbb{I}[1/3 < \tilde{\omega}][\tilde{\omega}(1/2)^n + \mathbb{I}[1/3 \geq \tilde{\omega}](-1/3 + 2\tilde{\omega})(1/3)^n], & \tilde{\omega} \leq 1/2, \\
\mathbb{I}[2/3 > \tilde{\omega}][\tilde{\omega}(1/2)^n + \mathbb{I}[2/3 \leq \tilde{\omega}](-2/3 + 2\tilde{\omega})(2/3)^n], & \tilde{\omega} > 1/2,
\end{cases}
\]

such that

\[
E \left[ \tilde{q}^\omega(p, y) \left( \frac{\partial}{\partial y} \tilde{q}^\omega(p, y) \right)^n \right] = (1/3)^n \int_0^{1/3} (-1/3 + 2\tilde{\omega}) + (1/2)^n \int_{1/3}^{1/2} \tilde{\omega}
\]

\[
+ (1/2)^n \int_{1/2}^{2/3} \tilde{\omega} + (2/3)^n \int_{2/3}^{1} (-2/3 + 2\tilde{\omega})
\]

\[
= 1/6(1/2)^n + 1/3(2/3)^n.
\]

Expressions (17) and (18) are only equal for \( n = 1 \). Since two observationally equiva-
lent models generate different results for \( n \geq 2 \), \( \{\mathbb{E}[q^{\omega}(p, y)(\frac{\partial}{\partial y} q^{\omega}(p, y))^n]\}_{n=2}^\infty \) are not nonparametrically identified.

\[ \]  

**D.1 Results for a finite population**

Suppose a population consists of only finitely many people \( \{\omega_1, \ldots, \omega_n\} \) with associated demand functions \( \{q^{\omega_1}, \ldots, q^{\omega_n}\} \). For simplicity, consider the two-goods case. Analogous to the setting with an infinite population, we can define the \( n \)th population moment as

\[
M_n(p, y) = \sum_{i=1}^{n} q^{\omega_i}(p, y)^n.
\]

Define the \( n \)th moment of the compensating variation as

\[
\mathbb{E}[CV^{\omega}(p_0, p_1, y)^n] = \frac{1}{n} \sum_{i=1}^{n} \left( e^{\omega_i}(p_1, v^{\omega_i}(p_1, y)) - e^{\omega_i}(p_0, v^{\omega_i}(p_1, y)) \right)^n
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( y - e^{\omega_i}(p_0, v^{\omega_i}(p_1, y)) \right)^n.
\]

The following two results also hold for this finite population.

**Theorem 7.** Suppose Assumption 1 holds. Then the second-order approximation of the \( n \)th moment of the compensating variation only depends on the \( n \)th and \((n + 1)\)th conditional moment of demand. It can be written as

\[
\mathbb{E}[CV^{\omega}(p_0, p_1, y)^n] = (\Delta p)^n \left( M_n(p, y_1) + \frac{\Delta p}{2} \left[ \frac{\partial}{\partial p} M_n(p, y_1) + \frac{n}{n+1} \frac{\partial}{\partial y} M_{n+1}(p, y_1) \right] + O((\Delta p)^2) \right).
\]

**Theorem 8.** If a moment sequence is rationalizable by a population of finite consumers, any polynomial which is positive in the support of demand at a given price must have a negative translation.

**Proof.** The proof is identical to the one we used for the continuum, replacing integrals with sums.

**E Empirical Application**

**E.1 Construction of the Categories**

Table 3 details on the basis of which COICOP codes the categories and their price indices were constructed. As stated in the main text, the category “other nondurables and services” consists of expenditures on alcohol and tobacco (02), clothing and footwear.
(03), communication (08), recreation (09), and restaurants and hotels (11). To construct the Stone-Lewbel price indices, we consider the following nested goods:

- **Food**: food (01.1) and non-alcoholic beverages (01.2)
- **Housing**: actual rentals for housing (04.1), maintenance, repair and security of the dwelling (04.3), water supply and miscellaneous services relating to the dwelling (04.4), and electricity, gas and other fuels (04.5)
- **Transportation**: purchase of vehicles (07.1), operation of personal transport equipment (07.2), passenger transport services (07.3)

### Table 3: Construction of the categories and their price indices

<table>
<thead>
<tr>
<th>Category</th>
<th>COICOP classification</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Expenditures</strong></td>
<td>Price index</td>
</tr>
<tr>
<td>food</td>
<td>01, 01.1, 01.2</td>
</tr>
<tr>
<td>housing</td>
<td>04, 04.1, 04.3, 04.4, 04.5</td>
</tr>
<tr>
<td>transportation</td>
<td>07, 07.1, 07.2, 07.3</td>
</tr>
<tr>
<td>other nondurables and services</td>
<td>02, 03, 08, 09, 11</td>
</tr>
</tbody>
</table>

### E.2 Descriptive Statistics

Table 4 provides descriptive statistics of the final sample that is used in the estimation of the conditional moments and the EASI demand system.

### Table 4: Descriptive statistics estimation sample (N = 12,494)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Income and expenditures (2015 prices)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>disposable income</td>
<td>529.13</td>
<td>287.13</td>
<td>91.07</td>
<td>1,637.97</td>
</tr>
<tr>
<td>food</td>
<td>55.60</td>
<td>32.08</td>
<td>1.82</td>
<td>292.78</td>
</tr>
<tr>
<td>housing</td>
<td>134.13</td>
<td>71.04</td>
<td>4.80</td>
<td>573.89</td>
</tr>
<tr>
<td>transportation</td>
<td>59.03</td>
<td>59.11</td>
<td>0.02</td>
<td>554.58</td>
</tr>
<tr>
<td>other nondurables and services</td>
<td>135.74</td>
<td>101.43</td>
<td>6.60</td>
<td>771.48</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variable</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Demographic characteristics</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>number of adults</td>
<td>1.85</td>
<td>0.76</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>number of children</td>
<td>0.77</td>
<td>1.08</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>number of retired</td>
<td>0.03</td>
<td>0.18</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>number of earners</td>
<td>1.24</td>
<td>0.89</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>
E.3 AIDS Model and Welfare Deviation

Under the ALmost Ideal Demand System (AIDS) (Deaton and Muellbauer, 1980), which is commonly used in empirical analysis, we have

\[ q_i^*(p, y) = \frac{y}{p_i} \left[ \alpha_i^\omega + \sum_{j=1}^{k} \gamma_{ij}^\omega \ln p_j + \beta_i \ln \left( \frac{y}{P} \right) \right] \]

where there are \( j \) goods and \( q_i \) denotes the demand of the \( i \)th. Further \( P \) us the translog price index.

Under the above specification, we can compute \( \text{Cov} \left( \frac{\partial q}{\partial y}, q \right) \).

\[ E[q_i^*(p, y)] = \frac{y}{p_i} \left[ E[\alpha_i^\omega] + \sum_{j=1}^{k} E[\gamma_{ij}^\omega] \ln p_j + \beta_i \ln \left( \frac{y}{P} \right) \right] \]

Also;

\[
E \left[ \frac{\partial q^\omega(p, y)}{\partial y} \right] = \frac{1}{p_i} \left[ E[\alpha_i^\omega] + \sum_{j=1}^{k} E[\gamma_{ij}^\omega] \ln p_j + \beta_i \ln \left( \frac{y}{P} \right) \right] + \frac{\beta_i}{p_i}
\]

\[
E \left[ q^\omega(p, y) \frac{\partial q^\omega(p, y)}{\partial y} \right] = E \left[ q^\omega(p, y) \times \frac{q^\omega}{y} + \frac{\beta_i}{p_i} \right]
\]

\[
= \frac{1}{y} E \left\{ q^\omega(p, y) \right\}^2 + \frac{\beta_i}{p_i} E \left[ q^\omega(p, y) \right]
\]

\[
E \left[ \frac{\partial q^\omega(p, y)}{\partial y} \right] E[q_i^*(p, y)] = \frac{1}{y} E \left[ q^\omega(p, y) \right]^2 + \frac{\beta_i}{p_i} E \left[ q^\omega(p, y) \right]
\]

\[
\text{Cov} \left( \frac{\partial q}{\partial y}, q \right) = E \left[ q^\omega(p, y) \frac{\partial q^\omega(p, y)}{\partial y} \right] - E \left[ \frac{\partial q^\omega(p, y)}{\partial y} \right] E[q_i^*(p, y)]
\]

\[
= \frac{1}{y} \left\{ E \left[ \left\{ q_i^*(p, y) \right\}^2 \right] - E \left[ q_i^*(p, y) \right]^2 \right\}
\]

\[
= \frac{1}{y} \text{Var}(q_i(p, y))
\]

Which can easily be computed from the observed data.
Homotheticity

A similar observation can be made for homothetic preferences.

\[ q^\omega_i(p, y) = \alpha^\omega_i(p)y \]

\[ \frac{\partial q^\omega_i(p, y)}{\partial y} = \alpha^\omega_i(p) \]

This lets us compute the covariance

\[ \text{Cov}\left( q_i(p, y), \frac{\partial q_i(p, y)}{\partial y} \right) = \mathbb{E}\left[ q^\omega_i(p, y) \frac{\partial q^\omega_i(p, y)}{\partial y} \right] - \mathbb{E}\left[ \frac{\partial q^\omega_i(p, y)}{\partial y} \right] \mathbb{E}[q^\omega_i(p, y)] \]

\[ = \mathbb{E}[\{\alpha_i\}^2]y - \mathbb{E}[\alpha_i]^2y \]

\[ = \frac{1}{y} \text{Var}(q_i(p, y)) \]

These variances are now displayed in Figure 5.

F Bounds and The HN Approach

This section discusses the reduction in bounds that our approach offers to the HN approach. Hausman and Newey (2016) Show that if income effects are bounded, i.e.,

\[ \forall \ \omega \in \Omega \quad A \leq I E^\omega \leq B \]

We can say that

\[ \Delta p \int_0^1 q^\omega(p(t), y)e^{A\Delta pt}dt. \leq \mathbb{E}[CV] \leq \Delta p \int_0^1 q^\omega(p(t), y)e^{B\Delta pt}dt. \]

We can rewrite the LHS as

\[ \Delta p \int_0^1 [q^\omega(p(t), y)] [1 + A\Delta pt + O(\Delta p)^2] \]

\[ = \Delta p \int_0^1 [q^\omega(p(t), y)dt] + (\Delta p)^2A \int_0^1 [q^\omega(p(t), y)dt] + O(\Delta p)^3 \]

We can simplify the above bounds by carrying out integration by parts

\[ \int_0^1 [q^\omega(p(t), y)dt] = \int_0^1 [q^\omega(p(t), y)dt] - \int_0^1 \left[ \int_0^s \{q^\omega(p(s), y)ds\} \right] dt \]
This means that the Range of the HN bounds is

$$(A - B)(\Delta p)^2 \left[ \int_0^1 [q^\omega(p(t), y)dt] - \int_0^1 \left[ \int_0^s \{q^\omega(p(s), y)ds\} \right] dt \right] + O(\Delta p)^3$$

**Theoretical plausible maximum deviation**  Let

$$CV(X) = \Delta p \int_0^1 [q^\omega(p(t), y)dt] + (\Delta p)^2 X \int_0^1 [q^\omega(p(t), y)tdt]$$

Set $X = B = 0$

$$\frac{CV(A) - CV(B)}{CV(B)} = A\Delta p \times \left[ 1 - \frac{\int_0^1 \left[ \int_0^s \{q^\omega(p(s), y)ds\} \right] dt}{\int_0^1 [q^\omega(p(t), y)tdt]} \right]$$

Which is observable and close to half when price effects are small.

When $A = \frac{1}{p}$, we have

$$\frac{CV(A) - CV(B)}{CV(B)} = \frac{\delta p}{p} \left[ 1 - \frac{\int_0^1 \left[ \int_0^s \{q^\omega(p(s), y)ds\} \right] dt}{\int_0^1 [q^\omega(p(t), y)tdt]} \right] \approx \frac{1}{2} \frac{\Delta p}{p}$$

This reflects the bounds in the first-order approach via the HN approach and the reduction of the uncertainty we offer. If prices increase by 15% we reduce the uncertainty in the estimates by around 7.5 percent.
Figure 5: Scaled variance of demand by income level

(a) Food

(b) Housing

(c) Transportation