Twenty-fourth Annual Iowa Collegiate Mathematics Competition
Grinnell College, Saturday February 17, 2018

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The problems are listed in no particular order of difficulty. Each solution requires a proof or justification. Answers only are not enough. Calculators are allowed but certainly not required.

1. Prove that
\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2018} = \frac{1}{1010} + \frac{1}{1011} + \frac{1}{1012} + \frac{1}{1013} + \ldots + \frac{1}{2018}
\]

*Hint*: This is a special case of a more general result. For all positive integers n,
\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}
\]

2. For the positive integer n, let S(n) be the set of non-empty subsets of \{1, 2, 3,\ldots, n\}.
Let P(n) be the collection of the products of the elements of the sets in S(n).
Let R(n) be the collection of reciprocals of the members in P(n).
For a collection of numbers A, let T(A) be the sum of the members of A.

*For example, for n=2 we have:*
\[
S(2) = \{\{1\}, \{2\}, \{1,2\}\}; \quad P(2) = <1, 2, 2>; \quad R(2) = <1, \frac{1}{2}, \frac{1}{2}>; \quad T(P(2)) = 5; \text{ and } T(R(2)) = 2.
\]

Prove:

a) \(T(P(n)) = (n + 1)! - 1\)

b) \(T(R(n)) = n\)

3. Determine all one-to-one functions \(f: \mathbb{R} \rightarrow \mathbb{R}\) such that \(f(f(x) + y) = f(x + y) + f(1)\)

4. A disc is divided into 6 sectors by three intersecting diameters. One checker is placed in each sector. On a given move, two checkers are moved to neighboring sectors. Can all the checkers end up in one sector?

5. Determine the cubic polynomial \(P(x)\) for which \((x - 1)^2\) is a factor of \(P(x) - 1\), and \((x + 1)^2\) is a factor of \(P(x) + 1\).
Let P be a point in the first quadrant on the line \( y = x \). Let line L through P intersect the x-axis at point A with coordinates \((a,0)\) and the y-axis at point B with coordinates \((0,b)\). Let line L’ through P intersect the x-axis at point A’ with coordinates \((a’,0)\) and the y-axis at point B’ with coordinates \((0,b’)\). 

Prove that \[
\frac{ab}{a + b} = \frac{a’b’}{a’ + b’}
\]

Evaluate the integral \[
\int \frac{x^3}{(x^2+1)^3} \, dx
\]
in two ways:

a) by the substitution, \( u = x^2 + 1 \)

b) by the substitution \( x = \tan \theta \)

c) Explain why the results above are the same.

Several boys and girls were at a dance. During the evening, every boy danced with at least one girl, but no girl danced with every boy. Prove that there were two boys, \( B_1 \) and \( B_2 \), and two girls \( G_1 \) and \( G_2 \), for which \( B_1 \) danced with \( G_1 \) and \( B_2 \) danced with \( G_2 \), but \( B_1 \) did not dance with \( G_2 \) and \( B_2 \) did not dance with \( G_1 \).

Consider the expansion of \((a+b+c+d+e)^{10}\)

a) How many terms contain neither c’s, nor d’s, nor e’s?

b) How many terms contain either an a or a b (or both)?

a) Prove that, for positive integers \( n \), \[
\sqrt{n + 1} - \sqrt{n} < \frac{1}{2\sqrt{n}} < \sqrt{n} - \sqrt{n - 1}
\]

b) Prove that \[
\left\lfloor \sum_{i=1}^{1010^2} \frac{1}{\sqrt{i}} \right\rfloor = 2018
\]

Notation: \([x]\) represents the greatest integer less than or equal to \( x \).