Seventh Annual Iowa Collegiate Mathematics Competition

Iowa State University
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Problem 1. Side of a triangle.

If the area of an equilateral triangle is $\frac{3}{4}$, what is the length of each side?

Problem 2. Solve for x.

Find all real solutions of the equation $\left\lfloor x \right\rfloor^2 - 5\left\lfloor x \right\rfloor - 6 = 0$.

Here $\left\lfloor x \right\rfloor$ denotes, as usual, the greatest integer less than or equal to x.

Problem 3. Taylor’s Theorem.

According to Taylor’s Theorem, if $f$ is a twice differentiable function on an interval containing $a$ and $b$, with $a \neq b$, then there is a number $c$ between $a$ and $b$ such that

$$f(b) - f(a) = f'(a)(b-a) + \frac{f''(c)}{2}(b-a)^2.$$  

Express $c$ as simply as possible in terms of $a$ and $b$ if $f(x) = x^3$.

Problem 4. An irrational number.

Let $r$ and $s$ be positive rational numbers with $\sqrt{r}$ irrational. Prove that $\sqrt{r} + \sqrt{s}$ is irrational.

Problem 5. Multiplicative inverses.

Let $R$ be the ring of integers modulo 2001. For example, in $R$, $1900 + 125 = 24$ and $90^2 = 96$. 
(a) Determine whether the element 1334 has a multiplicative inverse in R, and if so, find it. If not show this.

(b) Do the same for the element 1333.

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**Problem 6. A harmonic identity.**

For each positive integer \( n \), let

\[
\frac{h(n)}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.
\]

Prove that for every integer \( n \geq 2 \),

\[
n + h(1) + h(2) + \cdots + h(n-1) = nh(n).
\]

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**Problem 7. Sum of squares divisible by \( n \).**

A certain set of \( n \) integers has the property that the difference between the product of any \( n - 1 \) of them and the remaining one is divisible by \( n \). Prove that the sum of the squares of all \( n \) integers is divisible by \( n \).

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**Problem 8. Sum the series.**

Find the sum of the series

\[
\sum_{n=1}^{\infty} \frac{n^2 + 3n}{3^n},
\]

and justify your answer.

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**Problem 9. A multiple of 49.**

After several applications of the operation of differentiation and the operation of multiplication by \( x - 1 \), performed in unspecified order, the polynomial \( x^8 + x^7 \) is changed to \( ax + b \), where \( a \neq 0 \).

Prove that \( a - b \) is an integer divisible by 49.

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**Problem 10. Integer roots.**

Find all real numbers \( p \) such that all three roots of the cubic equation

\[
5x^3 - 5(p + 1) x^2 + (71p - 1)x + 1 = 66p
\]

are positive integers.