# Dynamics on character varieties and Hodge theory

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joint with Aaron Landesman &
Daniel Litt.

# Outline

# Background

## Definition

Define $\mathfrak{X}_n$ be the set of $n$-tuples of $2 \times 2$ matrices $(A_1, \cdots, A_n)$ such that

$A_i \in SL_2(\mathbb{C})$

$$A_1 \cdots A_n = \mathrm{Id},$$

considered up to simultaneous conjugation, i.e.

$$(A_1, \cdots, A_n) \sim (gA_1g^{-1}, \cdots gA_ng^{-1}),$$

for all $g \in \mathrm{SL}_2(\mathbb{C})$.

# Braid group action

$\boxed{A_1 A_2 \cdots A_n = \mathrm{id}}$

- For $i = 1, \cdots, n-1$, consider the maps $\sigma_i : \mathfrak{X}_n \to \mathfrak{X}_n$ given by

$$(A_1, \cdots, A_i, A_{i+1}, \cdots, A_n) \mapsto (A_1, \cdots, A_i A_{i+1} A_i^{-1}, A_i, \cdots, A_n).$$

$\mathfrak{X}_n$

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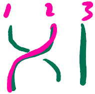
- The $\sigma_i$'s generate an action of the Artin braid group $B_n$ on $\mathfrak{X}_n$ :

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$

braid relation

and $\sigma_i \sigma_j = \sigma_j \sigma_i$ whenever $i \neq j \pm 1$.

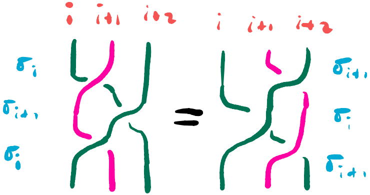
- The braid group $B_n$ parametrizes "braids" on $n$ strands.




$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } i \neq j \pm 1 \end{array} \right\rangle$$

$$\parallel$$

$$B_n$$



# Main result–vague version

braid gp

Upshot so far: action of $B_n$ on $\mathfrak{X}_n$, where each element of the latter is represented by tuple of $2 \times 2$-matrices $(A_1, \cdots, A_n)$, $A_i \in \mathrm{SL}_2(\mathbb{C})$.

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**Theorem (L.-Landesman-Litt)**

*Assume some $A_i$ has infinite order. Then we can classify explicitly all finite orbits of the $B_n$-action on $\mathfrak{X}_n$.*

# Infinitely many examples

$n=4$ , $B_4 \curvearrowright \mathcal{X}_4$

## Example

$$A_1 = \begin{pmatrix} 1 + x_2x_3/x_1 & -x_2^2/x_1 \\ x_3^2/x_1 & 1 - x_2x_3/x_1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix},$$

$A_1A_2A_3A_4 = \mathrm{id}$

where

$$x_1 = 2\cos\left(\frac{\pi(\alpha+\beta)}{2}\right), x_2 = 2\sin\left(\frac{\pi\alpha}{2}\right), x_3 = 2\sin\left(\frac{\pi\beta}{2}\right),$$

for any $\alpha, \beta \in \mathbb{Q}$.

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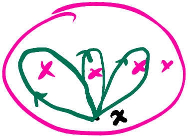
Can check that these indeed give finite orbits of $B_4$...

# Motivation (from algebraic geometry)



Consider the Riemann surface $X := \mathbb{P}^1 \setminus \{x_1, \cdots, x_n\}$ for distinct $x_i \in \mathbb{C}$.

## Observation

Fix a basepoint $x \in X$. A representation $\rho : \pi_1(X, x) \to \mathrm{SL}_2$ is precisely a tuple $(A_1, \cdots, A_n)$ such that $A_1 \cdots A_n = \mathrm{Id}$. These are also known as local systems on $X$.

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Therefore, $\mathfrak{X}_n$ is the set of isomorphism classes of $\rho : \pi_1(X) \to \mathrm{SL}_2$.

rank 2 local systems on $X$

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## Observation

$\mathfrak{X}_n$ is the set of isomorphism classes of $\rho : \pi_1(X) \to \mathrm{SL}_2$.

"$\mathbb{P}^1 \setminus \{x_1, \dots, x_n\}$"

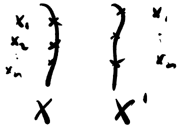
Vary $X$ in moduli, i.e. move the points $x_1, \cdots, x_n$ around. For $X'$ near to $X$, we also get a representation $\pi_1(X') \to \mathrm{SL}_2$.

$B_n$ IS

Going around loops in the moduli space $\mathcal{M}_{0,n}$, get an action of $\pi_1(\mathcal{M}_{0,n})$ on $\mathfrak{X}_n$, same as before.

$\sigma_i \in \pi_1(\mathcal{M}_{0,n})$

which element is this?




moduli of n-punctured $\mathbb{P}^1$'s $= \mathcal{M}_{0,n}$

isomorphism class of local system on $X$

The finite orbits of the $B_n$-action on $\mathfrak{X}_n$ therefore correspond to certain special local systems–*MCG (mapping class group) finite*, or *canonical*.

braid gp for $\mathcal{M}_{0,n}$

**Remark 1**

Trivial examples: those with monodromy a finite group $G \subset \mathrm{SL}_2$.

$A_i \in G \ \forall i$.
$A_1, \ldots, A_{n-1}$

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### Remark 2

The same definition works for $\rho : \pi_1(\Sigma) \to \mathrm{GL}_n$, where $\Sigma$ is any Riemann surface. Very non-trivial examples: conformal blocks of e.g. Wess-Zumino-Witten CFT.

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### Question

Can we classify all MCG-finite local systems?

Recall: $X = \mathbb{P}^1 \setminus \{x_1, \cdots x_n\}$.

## Algebro-geometric reformulation of MCG-finiteness

A local system $\mathbb{V}$ on $X$ is MCG-finite if and only if there exists a diagram

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \mathcal{C}_{0,n} \arrow[d] & \text{universal } n\text{-punctured } \mathbb{P}^1 \\
p \arrow[r, hook] & \widetilde{\mathcal{M}} \arrow[r, "\pi", "\text{finite étale}"'] \arrow[ur, phantom, "\lrcorner"] & \mathcal{M}_{0,n} & \text{moduli of } n\text{-punctured } \mathbb{P}^1\text{'s}
\end{tikzcd}$$

such that $\mathbb{V}$ extends to a local system $\widetilde{\mathbb{V}}$ on $\widetilde{\mathcal{C}}$:

Prop this is equiv. to previous definition. $\square$

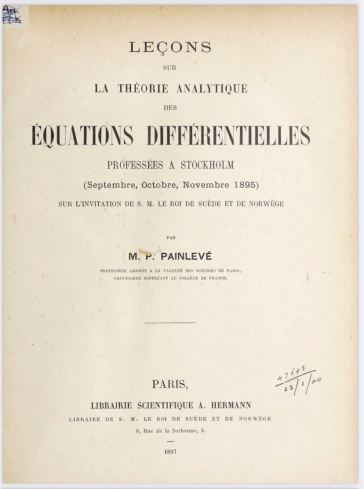
# Simplest non-trivial case of this question

$B_4 \curvearrowright \mathcal{X}_4$

When $n = 4$, the question is equivalent to finding all algebraic solutions of Painlevé VI. $y(t)$

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right\}$$

# History of PVI



LEÇONS

SUR

LA THÉORIE ANALYTIQUE

DES

ÉQUATIONS DIFFÉRENTIELLES

PROFESSÉES A STOCKHOLM

(Septembre, Octobre, Novembre 1895)

SUR L'INVITATION DE S. M. LE ROI DE SUÈDE ET DE NORWÈGE

PAR

M. P. PAINLEVÉ

PROFESSEUR ADJOINT A LA FACULTÉ DES SCIENCES DE PARIS,
PROFESSEUR SUPPLÉANT AU COLLÈGE DE FRANCE.

PARIS,

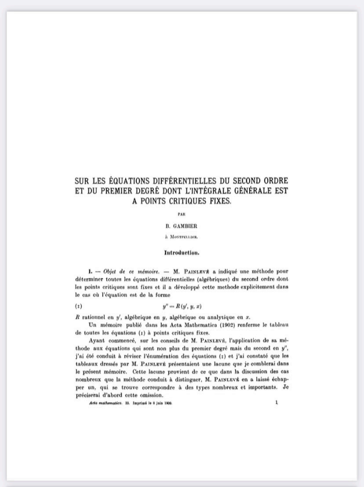
LIBRAIRIE SCIENTIFIQUE A. HERMANN

LIBRAIRE DE S. M. LE ROI DE SUÈDE ET DE NORWÈGE

8, Rue de la Sorbonne, 8

1897

(a) Painlevé's Stockholm lectures



SUR LES ÉQUATIONS DIFFÉRENTIELLES DU SECOND ORDRE ET DU PREMIER DEGRÉ DONT L'INTÉGRALE GÉNÉRALE EST A POINTS CRITIQUES FIXES.

PAR

B. GAMBIER

à Montpellier.

**Introduction.**

**1.** — *Objet de ce mémoire.* — M. Painlevé a indiqué une méthode pour déterminer toutes les équations différentielles (algébriques) du second ordre dont les points critiques sont fixes et il a développé cette methode explicitement dans le cas où l'équation est de la forme

$$y'' = R(y', y, x) \tag{1}$$

$R$ rationnel en $y'$, algébrique en $y$, algébrique ou analytique en $x$.

Un mémoire publié dans les Acta Mathematica (1902) renferme le tableau de toutes les équations (1) à points critiques fixes.

Ayant commencé, sur les conseils de M. Painlevé, l'application de sa méthode aux équations qui sont non plus du premier degré mais du second en $y''$, j'ai été conduit à réviser l'énumération des équations (1) et j'ai constaté que les tableaux dressés par M. Painlevé présentaient une lacune que je comblerai dans le présent mémoire. Cette lacune provient de ce que dans la discussion des cas nombreux que la méthode conduit à distinguer, M. Painlevé en a laissé échapper un, qui se trouve correspondre à des types nombreux et importants. Je préciserai d'abord cette omission.

*Acta mathematica.* 33. Imprimé le 8 juin 1909. 1

(b) Gambier's work on Painlevé equations

Figure 1

# History of PVI

- Painlevé studied several non-linear ODEs with a special property "only movable singularities are poles". Due to a calculation error, hiss list did not include PVI, which was later added in by his student Gambier; the solutions are known as Painlevé transcendants. ~ solns to Painlevé's eqns

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- Discovered independently by Richard Fuchs, whose point of view is the modern one: they govern isomonodromic deformations of linear ODEs.

Painlevé VI

(a) Paul Painlevé

(b) Richard Fuchs

# Main result–precise version

**Theorem (L.-Landesman-Litt)**

*Suppose* $(A_1, \cdots, A_n)$ *corresponds to a MCG-finite rank two local system* $\mathbb{V}$ *on* $X = \mathbb{P}^1 \setminus \{x_1, \cdots, x_n\}$*, with an* $A_i$ *of infinite order, and* $\mathrm{SL}_2$*-monodromy. Then it is of one of two possible types:*

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1. *it is of pullback type, i.e. for all* $X \in \mathcal{M}_{0,n}$, *the local system is* $f^*\mathbb{V}'$ *where* $f : X \to D$, $D$ *is a* ***fixed*** *curve, and* $\mathbb{V}'$ *a* ***fixed*** *local system on* $D$

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   can relax.

   middle convolution

2. *$\mathbb{V}$ is of the form $MC_\chi(\mathbb{U})$, where $\mathbb{U}$ is a local system of finite monodromy on $X$, with monodromy a complex reflection group.*

(1)

$$\begin{array}{ccccccc} X & \hookrightarrow & \tilde{\mathcal{E}} & \to & \mathcal{C}_{0,n} & \hookleftarrow & X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P & \in & \tilde{\mathcal{M}} & \to & \mathcal{M}_{0,n} & \ni & P' \end{array}$$

$\mathbb{V} \Rightarrow \tilde{\mathbb{V}}$

MCG finite

$X' \xrightarrow{f} D$

w/ fixed

(2) Katz's middle convolution

finite order character of $\mathbb{G}_m$

$MC_\chi$ : Local system on $X$ $\to$ Local system on $X$

- not rank preserving
- commute with $B_n$- action.

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2. $\mathbb{V}$ *is of the form* $MC_\chi(\mathbb{U})$, *where* $\mathbb{U}$ *is a local system of finite monodromy on* $X$, *with monodromy a complex reflection group.*

## Remark

The MCG-finite rank two local systems of pullback type are very constrained, and have been classified by Diarra–these only exist for $n \leq 6$. Also previously studied by Kitaev and Doran. (new algebraic soln to PVI)

# Main result-explanation

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## Definition

An finite order element of $A \in \mathrm{GL}_N(\mathbb{C})$ is a pseudo-reflection if $A - \mathrm{Id}$ has rank one. A finite complex reflection group $G \subset \mathrm{GL}_N(\mathbb{C})$ is a finite group generated by pseudo-reflections.

dihedral gp $\subset GL_2$ .

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**Theorem (L.-Landesman-Litt)** ($A_i$ has infinite order)

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Classified by Shephard-Todd: infinite family $G(m, p, n)$, and 34 exceptional ones, e.g. $W(E_8)$, icosahedral group, Valentiner's group...

## Example

Examples of MCG-finite local systems

$$A_1 = \begin{pmatrix} 1 + x_2x_3/x_1 & -x_2^2/x_1 \\ x_3^2/x_1 & 1 - x_2x_3/x_1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix},$$

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for any $\alpha, \beta \in \mathbb{Q}$.

Type 2 $\left(= MC_\chi(\underbrace{U}_{\text{dihedral monodromy}})\right)$

These examples correspond to dihedral groups inside $\mathrm{GL}_N(\mathbb{C})$, and are the same as those first constructed by Yang-Zuo. We also found several other examples, e.g. corresponding to icosahedral group $H_3 \subset \mathrm{GL}_3$.

# Previous work

There is a huge amount of literature on finding and classifying finite braid group orbits on $\mathfrak{X}_n$, especially the case of $n = 4$. In this case, $\mathfrak{X}_4$ may be written as

$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d.$$

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$$x^2 + y^2 + z^2 + xyz = ax + by + cz + d.$$

This surface (with parameters $a, b, c, d$ fixed) goes back to Fricke and Klein, and the dynamics was studied by Markoff. Various aspects of this dynamical system were studied by Goldman, Cantat-Loray, Bourgain-Gamburd-Sarnak, etc.
Algebraic solutions to PVI were found by Andreev-Kitaev, Boalch, Doran, Kitaev, Dubrovin-Mazzocco, Hitchin..., and recently shown to be complete by a computer search by Lisovvy-Tykhyy.

# Previous work on algebraic solutions to PVI

$n=4$

$A_1, A_2, A_3$ are all unipotent

$\sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

~30 pages

Invent. math. 141, 55–147 (2000)
Digital Object Identifier (DOI) 10.1007/s002220000065

Inventiones mathematicae

**Monodromy of certain Painlevé–VI transcendents and reflection groups**

B. Dubrovin, M. Mazzocco

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Oblatum 19-III-1999 & 25-XI-1999
Published online: 21 February 2000 – © Springer-Verlag 2000

**Abstract.** We study the global analytic properties of the solutions of a particular family of Painlevé VI equations with the parameters $\beta = \gamma = 0$, $\delta = \frac{1}{2}$ and $2\alpha = (2\mu - 1)^2$ with arbitrary $\mu, 2\mu \notin \mathbb{Z}$. We introduce a class of solutions having critical behaviour of algebraic type, and completely compute the structure of the analytic continuation of these solutions in terms of an auxiliary reflection group in the three dimensional space. The analytic continuation is given in terms of an action of the braid group on the triples of generators of the reflection group. We show that the finite orbits of this action correspond to the algebraic solutions of our Painlevé VI equation and use this result to classify all of them. We prove that the algebraic solutions of our Painlevé VI equation are in one-to-one correspondence with the regular polyhedra or star-polyhedra in the three dimensional space.

**Introduction**

In this paper, we study the structure of the analytic continuation of the solutions of the following differential equation

$$y_{xx} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y_x + \frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2 + \frac{x(x-1)}{(y-x)^2}\right], \qquad \text{PVI}\mu$$

where $x \in \mathbb{C}$ and $\mu$ is an arbitrary complex parameter satisfying the condition $2\mu \notin \mathbb{Z}$.

This is a particular case of the general Painlevé VI equation $\mathrm{PVI}(\alpha, \beta, \gamma, \delta)$, that depends on four parameters $\alpha, \beta, \gamma, \delta$ (see [Ince]).

(a) Dubrovin-Mazzocco

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ELSEVIER

**Algebraic solutions of the sixth Painlevé equation**

CrossMark

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ABSTRACT

We describe all finite orbits of an action of the extended modular group $\bar{\Lambda}$ on conjugacy classes of $SL_2(\mathbb{C})$-triples. The result is used to classify all algebraic solutions of the general Painlevé VI equation up to parameter equivalence.

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**1. Introduction**

Modular group $\Gamma = PSL_2(\mathbb{Z})$ consists of $2 \times 2$ matrices with integer entries and unit determinant, considered up to overall sign. It has a presentation $\Gamma = \langle s, t \mid s^3 = t^2 = 1 \rangle$, and is known to be isomorphic to the quotient of 3-braid group $\mathcal{B}_3$ by its center $\mathcal{Z} \cong \mathbb{Z}$. The kernel of the canonical homomorphism $\Gamma \to PSL_2(\mathbb{Z}_2) \cong S_3$ defines a congruence subgroup $\Lambda \subset \Gamma$, also known as $\Gamma(2)$:

$$\Lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a, d \text{ odd};\ b, c \text{ even} \right\} / \{\pm 1\}.$$

There are isomorphisms $\Lambda \cong \mathcal{P}_3/\mathcal{Z} \cong \mathcal{F}_2$, where $\mathcal{P}_3$ denotes the group of pure 3-braids and $\mathcal{F}_2$ is the free group with 2 generators.

Extended modular groups $\bar{\Gamma}$ and $\bar{\Lambda}$ are obtained by replacing the unit determinant condition with $ad - bc = \pm 1$. These groups have the following presentations:

$$\bar{\Gamma} = \langle r, s, t \mid r^2 = s^3 = t^2 = (tr)^2 = (sr)^2 = 1 \rangle, \tag{1}$$

$$\bar{\Lambda} = \langle x, y, z \mid x^2 = y^2 = z^2 = 1 \rangle \cong C_2 * C_2 * C_2, \tag{2}$$

where

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$x = rsts = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \quad y = rt = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z = stsr = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}.$$

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(b) Lisovyy-Tykhyy

# Digression: more algebraic geometry

## Definition

A local system $\mathbb{V}$ is of geometric origin if there exists a family of (smooth proper) varieties $\pi : Y \to X$, and $i$, such that $\mathbb{V}$ is a summand of $R^i\pi_*\underline{\mathbb{C}}$

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## Corollary

*We can classify all rank two local systems on the generic* $\mathbb{P}^1 \setminus \{x_1, \cdots, x_n\}$ *of geometric origin, assuming one local monodromy* $A_i$ *has infinite order.*

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## Remark

Again, our classification splits into two types: pullback solutions, and middle convolution.

# So, what is middle convolution?

$MC_\chi$

- Invented by Katz in 90's
- Some kind of kernel transforms, analogous to Fourier transform
- Fancy terms: $\exists$ an exotic tensor product on $Shv(\mathbb{G}_m)$
- if $U$ is finite monodromy on $X$, $MC_\chi(U)$ is

$X = \mathbb{P}^1 - \{x_1, \dots, x_n\} \hookrightarrow \mathbb{G}_m$

# Sketch of proof of main Theorem

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This implies $\mathbb{U} := MC_\chi(\mathbb{V})$ has finite monodromy, as the monodromy of the sum $\bigoplus_\sigma \mathbb{U}^\sigma$ is discrete and contained in a compact (unitary) group. Finally, can analyze the local monodromies to see that the finite group must be a complex reflection group.

# A few words on the condition of infinite local monodromy

- By Corlette-Simpson, our MCG-finite local system underlies a ($\mathbb{C}$)-VHS, — variation of Hodge Structure.
  and the same for all Galois conjugates.

- In rank 2, type of VHS is either

  (1,1) or (2,0) (or (0,2))

  repn into $SL_2(\mathbb{R})$ — repn into $SU_2$ (unitary)

- Infinite monodromy $\Rightarrow$ all of type (1,1).

- $MC_\chi$ has the effect of turning
  (1,1) $\rightsquigarrow$ unitary.
  (2,0) $\not\rightsquigarrow$ unitary

# Conclusions and open questions

## Upshot

We classified all rank two MCG-finite local systems on $\mathbb{P}^1 \setminus \{x_1, \cdots, x_n\}$, assuming one of the $A_i$'s has infinite order.

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- Our results recover those of Dubrovin-Mazzocco, and gives almost a conceptual classification of algebraic solutions to PVI.
- Our result can also be thought of as the classification rank two representations of certain finite index subgroups of the braid group $B_{n+1}$.
- Some combinatorics left to do, to analyze all complex reflection groups...

# Conclusions and open questions

- How to get rid of the condition of $A_i$ having infinite order? Suffices to show: any rank two MCG-finite local system on $\mathbb{P}^1 \setminus \{x_1, \cdots, x_n\}$, such that every simple closed loop has finite order, is of finite monodromy.

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- Higher rank local systems?
- We studied MCG-finite points here. What about MCG-finite subvarieties of $\mathfrak{X}_n$?

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[d] & \mathcal{C} \arrow[d] & \\ \tilde{M} \arrow[r, "\pi"] & M_{0,4} \arrow[r, phantom, "="] & \mathbb{P}^1 - \{0,1,\infty\} \end{tikzcd}$$

Thank you for your attention!

$X = \mathbb{P}^1 - \{x_1, \dots, x_n\}$

$\mathbb{V}$ rank 2.

$\mathbb{V}$ is never rigid on $X$ if $n \geq 4$.

- Katz MC reduces rank of rigid local system