Lecture 4
Wednesday, June 25, 2025 12:49 PM

Last Time

Orientability]
Euler Characteristic] Invariants

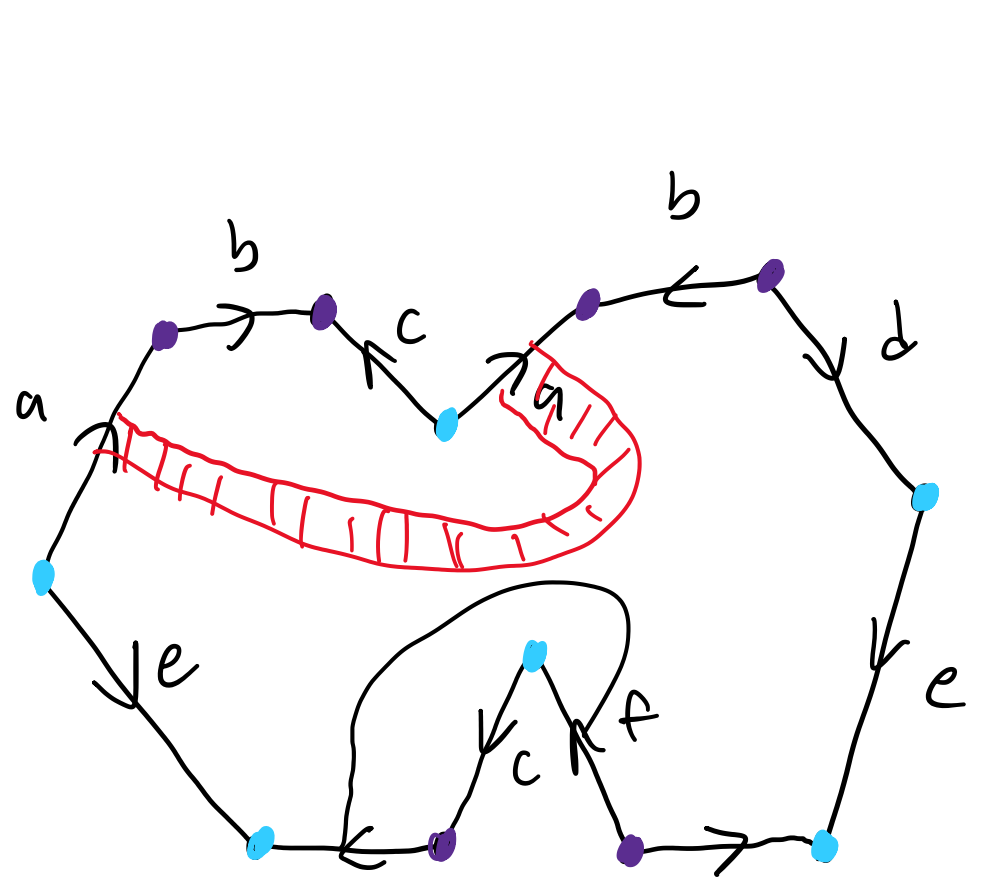
Today

Invariants
Surfaces w/ boundary
Knots

Riddle

When you look in the mirror, why are left and right switched but up and down are not?

Ex] I identify the 2-manifold below



$V=2$
$E=6$
$F=1$

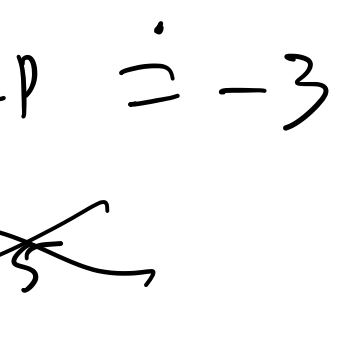
→ Nonorientable
$\chi = V-E+F = -3$
$\chi(\#^q \mathbb{R}P^2) = 2-q$ $\quad q=5$
$M \cong \#^5 \mathbb{R}P^2$

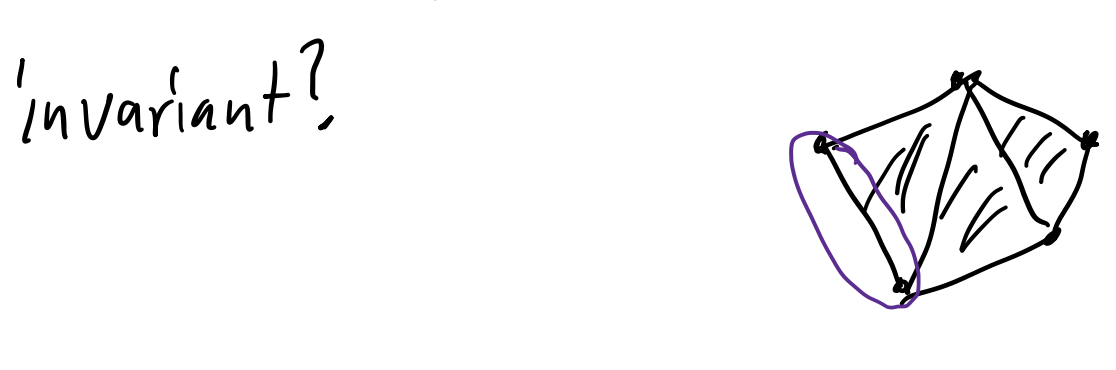
Q: What if I told you $\chi(M) = \underline{\underline{-3.}}$ What can you say about M?

$\chi(\#^p T^2) = 2-2p \overset{?}{=} -3$
~~$p = 2.5$~~

$\Rightarrow \#^5 \mathbb{R}P^2$

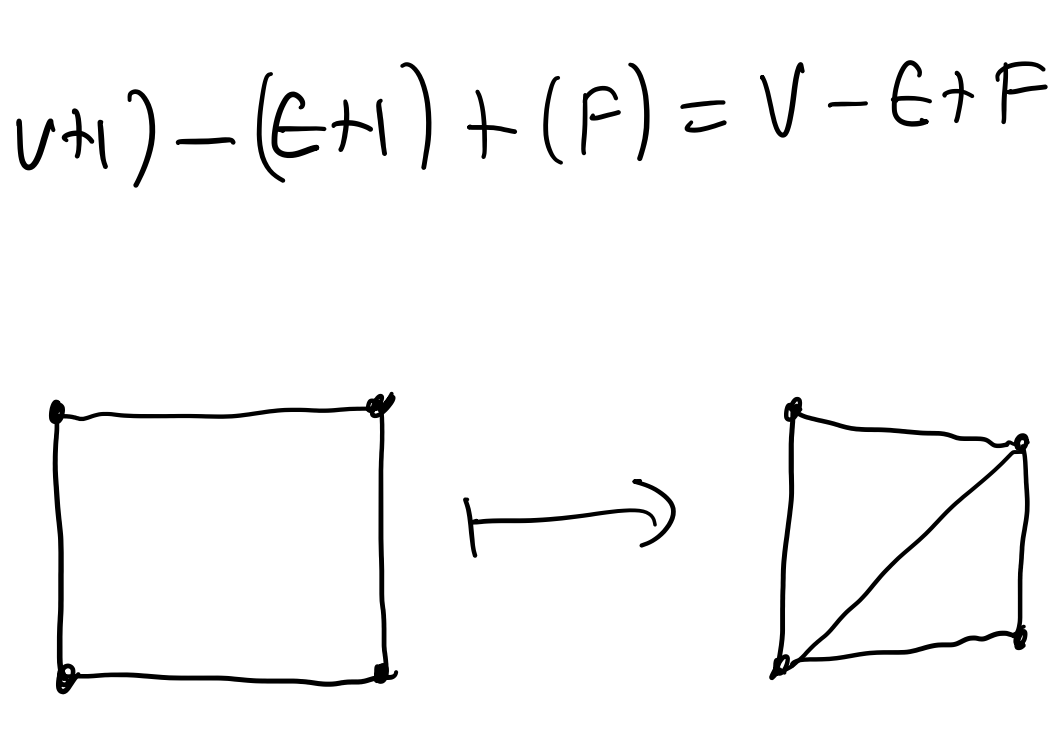
Q: Why is $\chi = V-E+F$ an invariant?



(A)


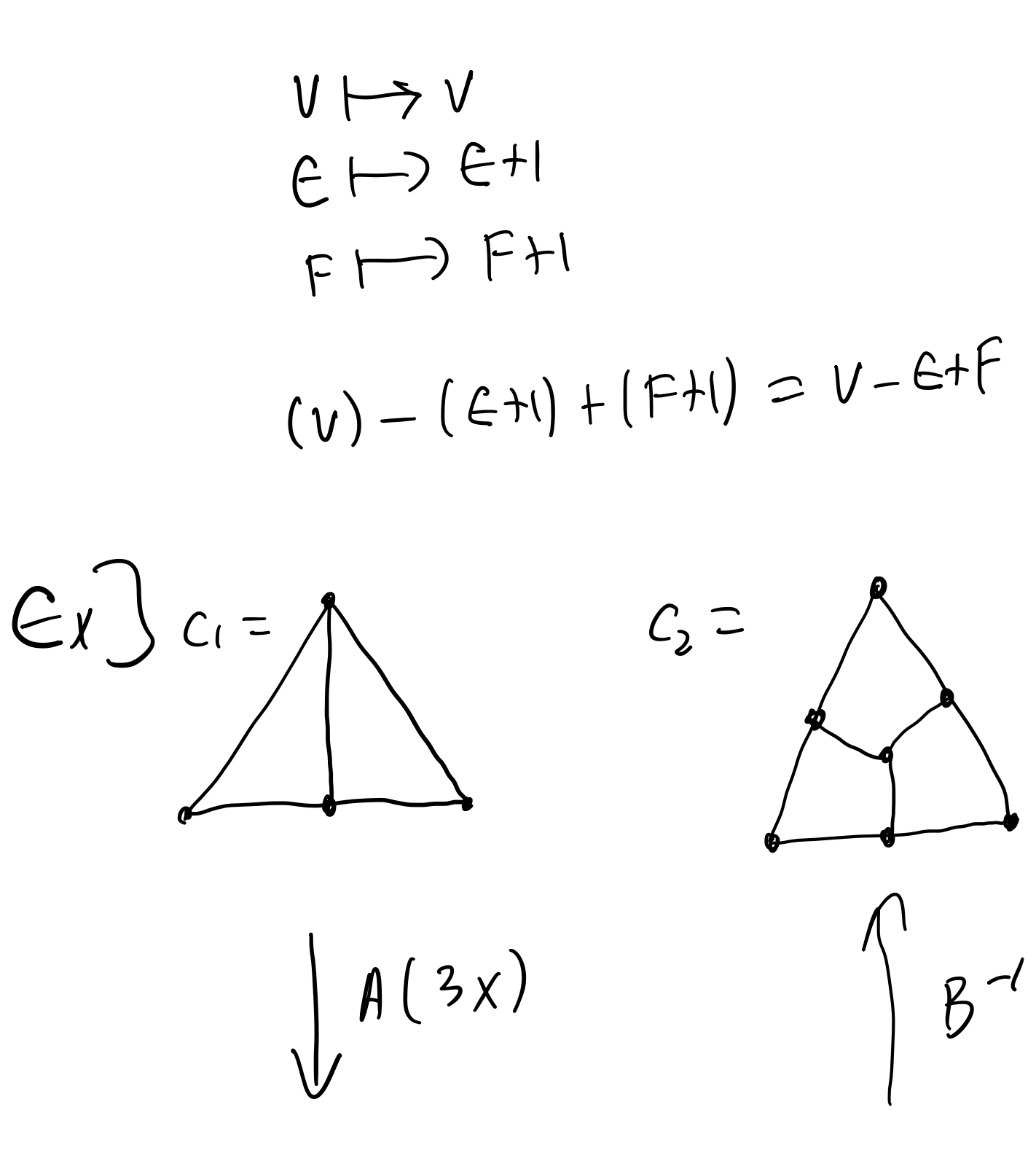
$V \longmapsto V+1$
$E \longmapsto E+1$
$F \longmapsto F$

$(V+1)-(E+1)+(F) = V-E+F$

(B)


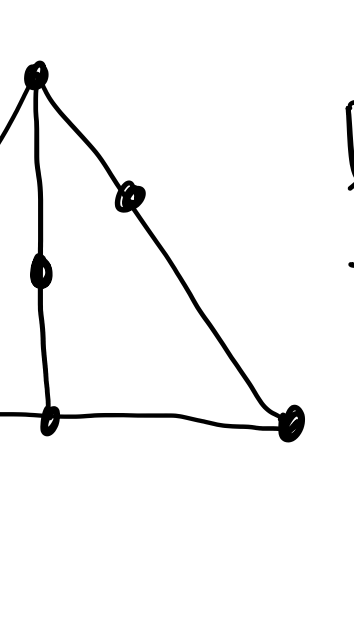
$V \longmapsto V$
$E \longmapsto E+1$
$F \longmapsto F+1$

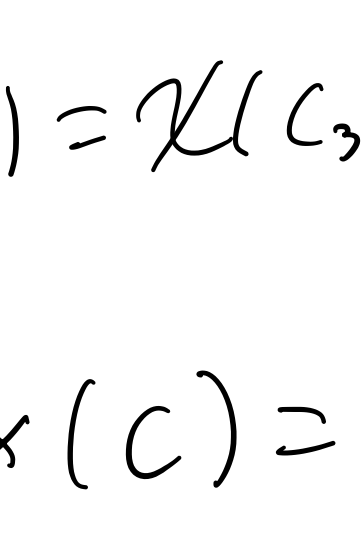
$(V)-(E+1)+(F+1) = V-E+F$

Ex]


$\chi(C_1) = \chi(C_3) = \chi(C_4) = \chi(C_2)$

Ex] $\alpha(C) = V-E$

$\mathbb{D}^2 =$

$\alpha = 2-2 = 0$

$\mathbb{D}^2 =$

$\alpha = 2-3 = -1$

α not an invariant.

Ex] $\beta(C) \overset{\text{def}}{=} 10$

Is β an invariant? Yes!

I give you a manifold M.

$\beta(M) = 10$

Ex] $\gamma(M) = \min\left(\begin{matrix}\text{\# edges in a} \\ \text{polygon representing} \\ M\end{matrix}\right)$

$\gamma(T^2) = 4$
$\gamma(\mathbb{R}P^2) = 2$
$\gamma(T^2 \# T^2) = 8$

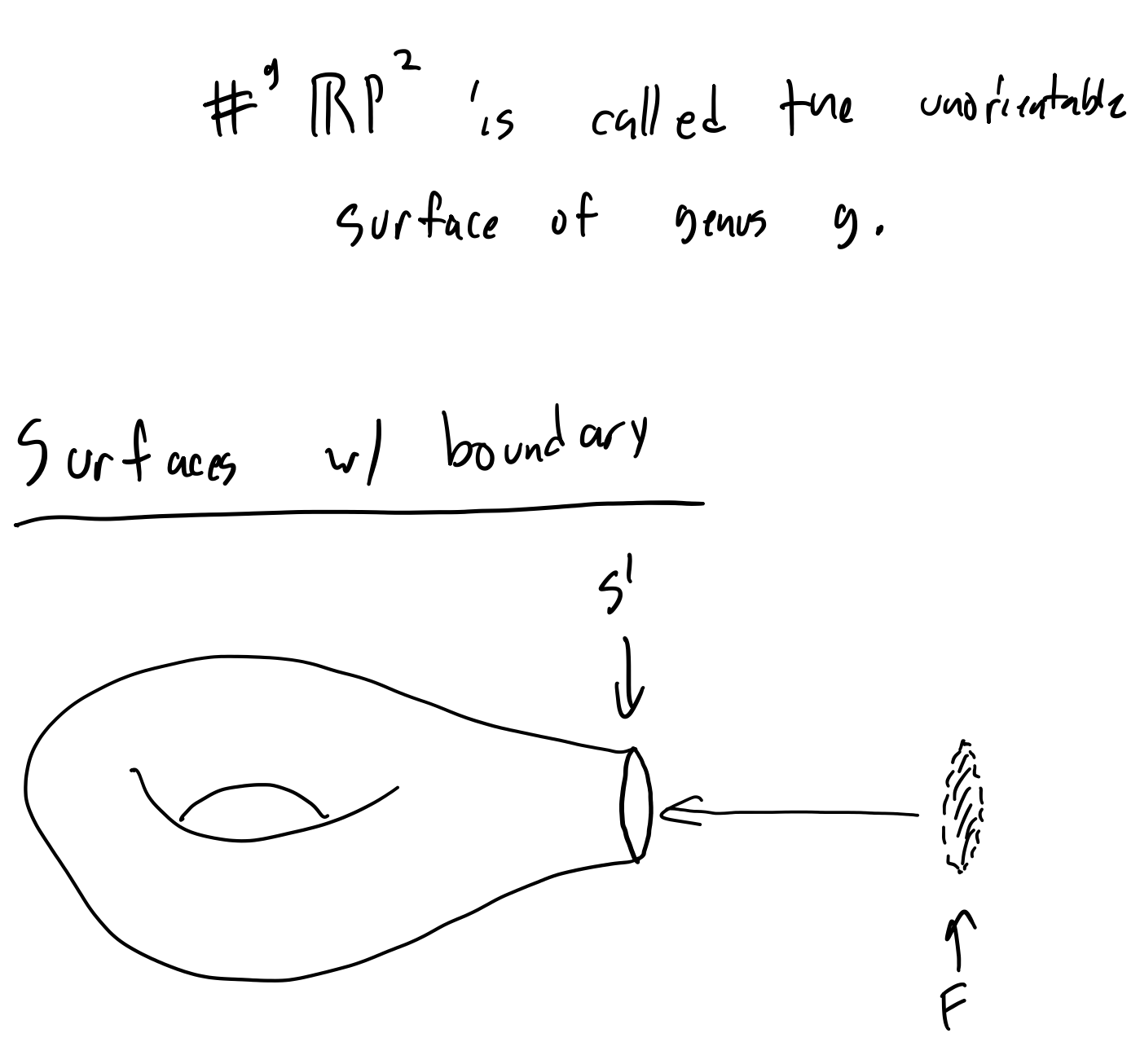
Invariant? ✓
Useful? ✓
Easy to compute? X

Terminology:

$\#^g T^2$ is called the (orientable) surface of genus g.

$\#^g \mathbb{R}P^2$ is called the unorientable surface of genus g.

Surfaces w/ boundary

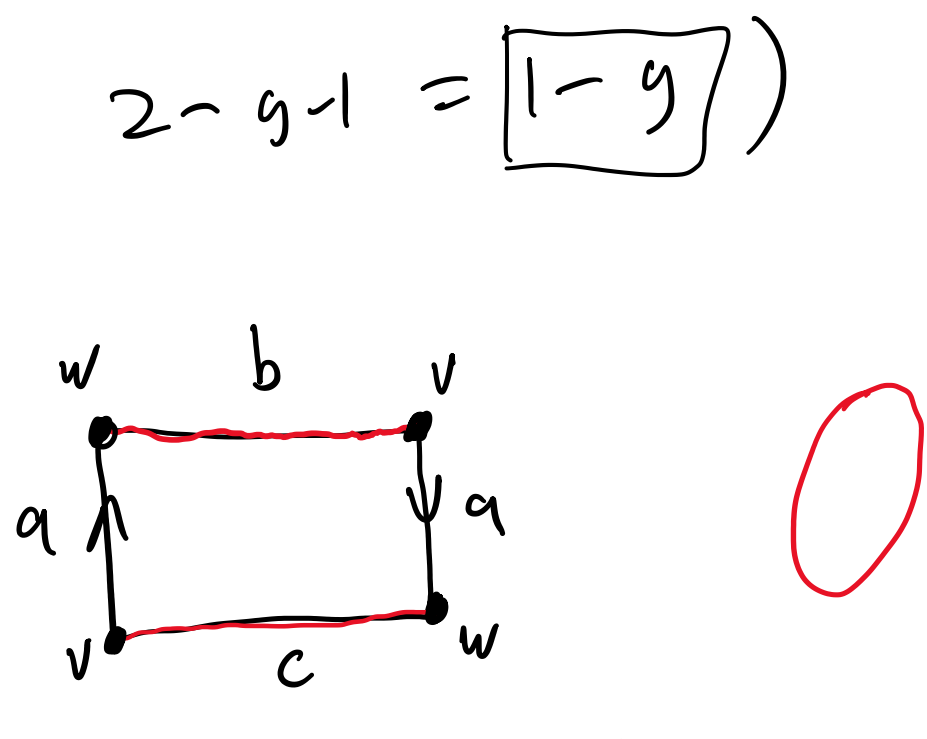


Def: The genus of a surface w/ boundary S^1 is the genus of the closed surface obtained by "filling in" the boundary w/ a disc

Theorem: The Euler characteristic of an orientable surface of genus g with one boundary component is

$2-2g-1 = 1-2g$

(For non-orientable, same but $2-g-1 = \boxed{1-g}$)

Ex]


$= \#^g \mathbb{R}P^2 - \mathbb{D}^2$

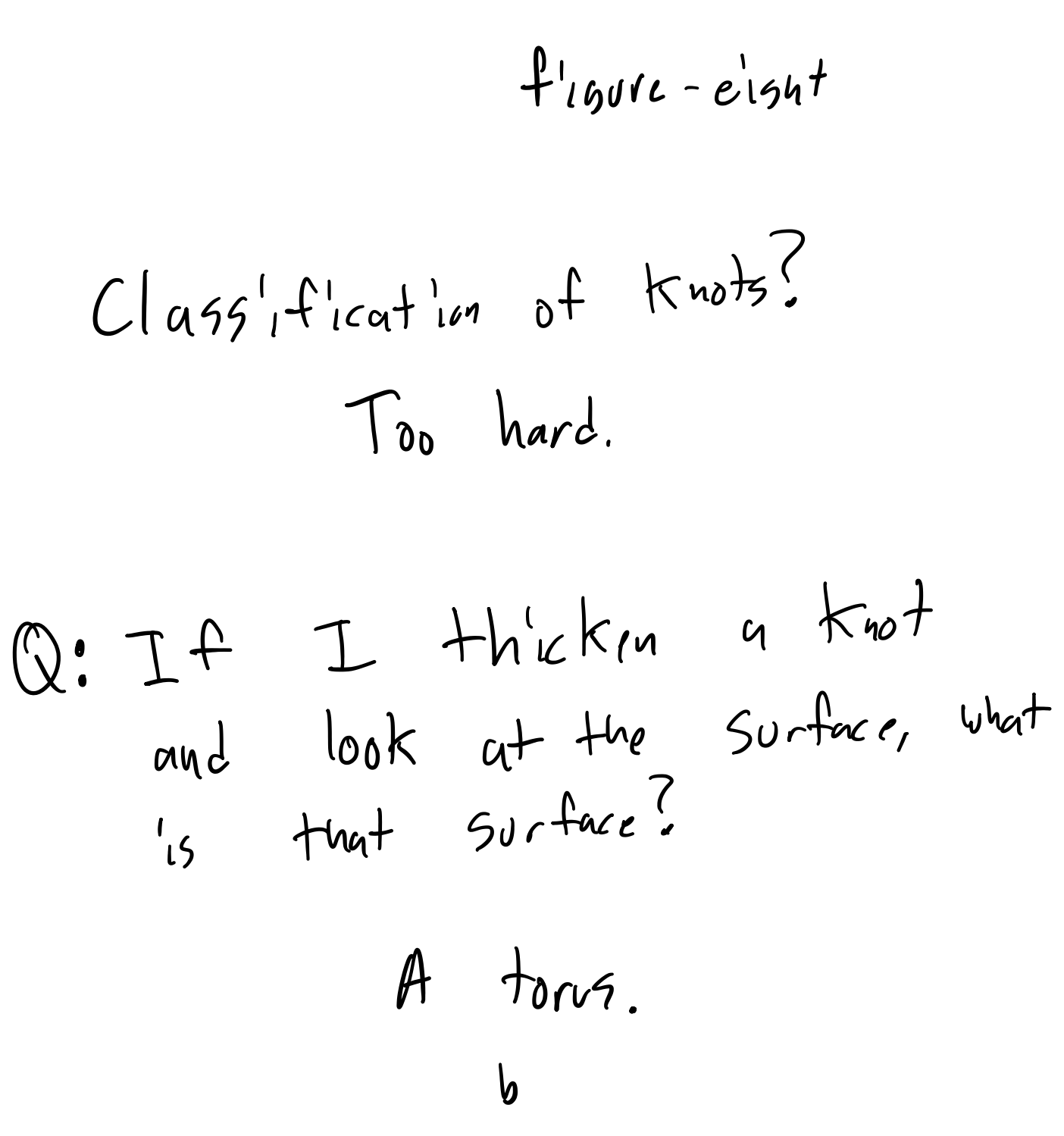
$V=2$ $E=3$ $F=1$

$\chi(\text{Möbius band}) = 0$

$1-g = 0$ $\quad g=1$

Möbius band $\cong \mathbb{R}P^2 - \mathbb{D}^2$

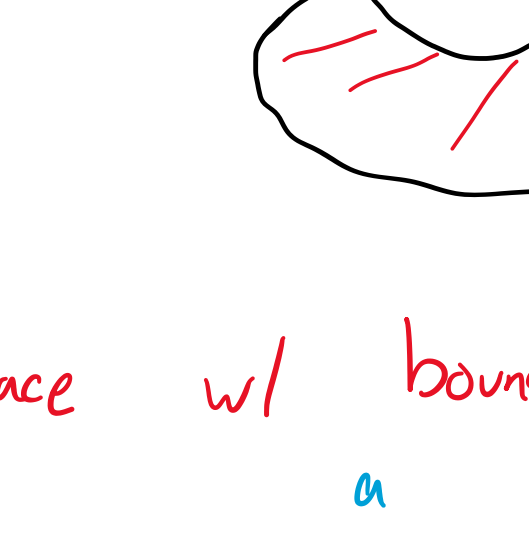
Knots: Circles in $\mathbb{R}^3$



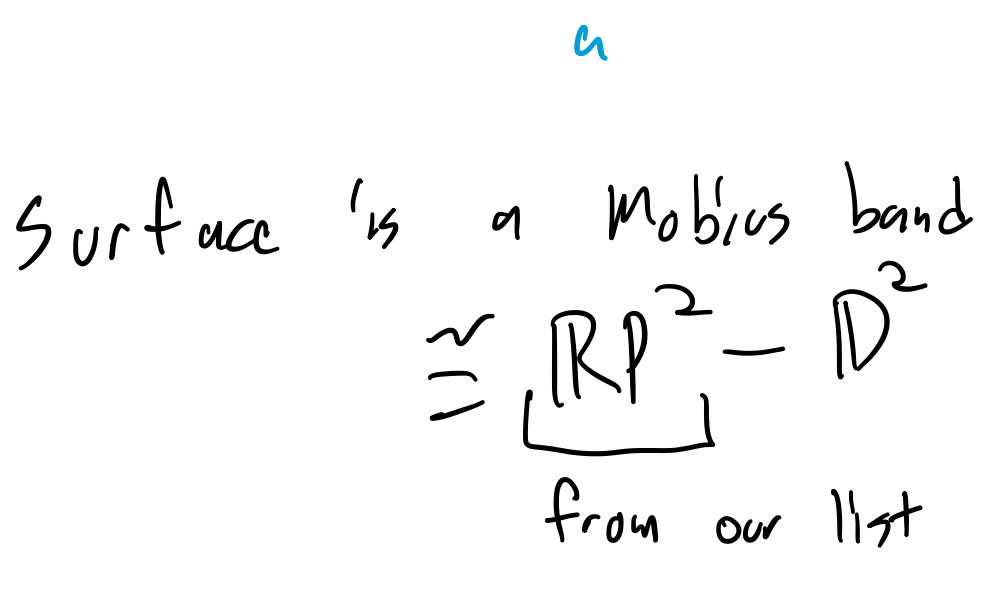
Classification of knots?
Too hard.

Q: If I thicken a knot and look at the surface, what is that surface?

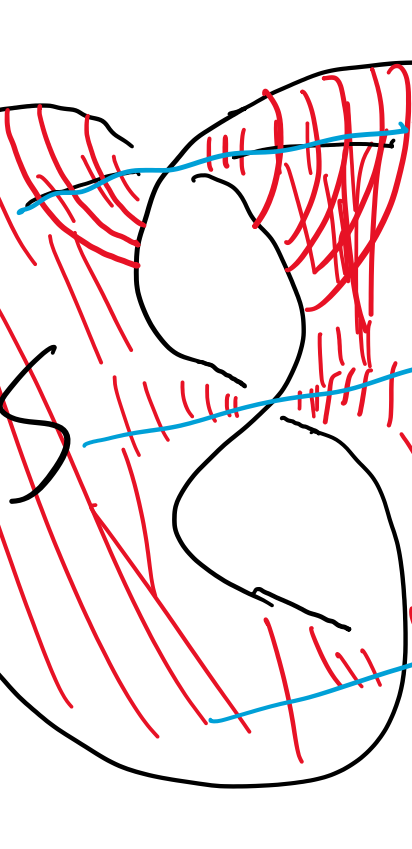
A torus.



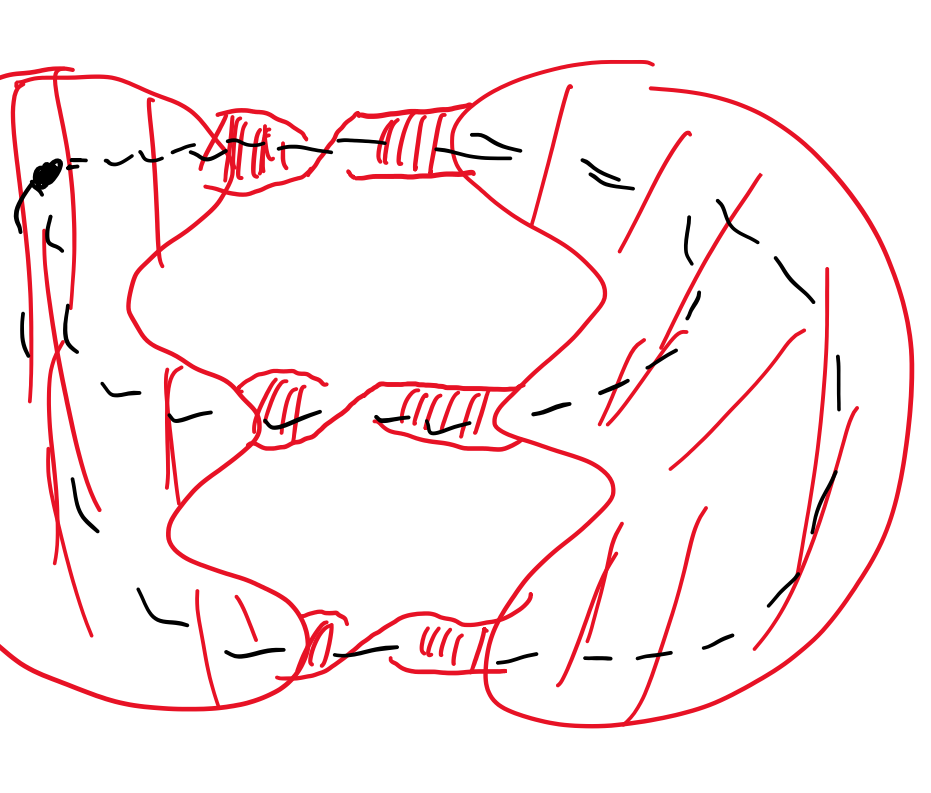
Ex] Trefoil



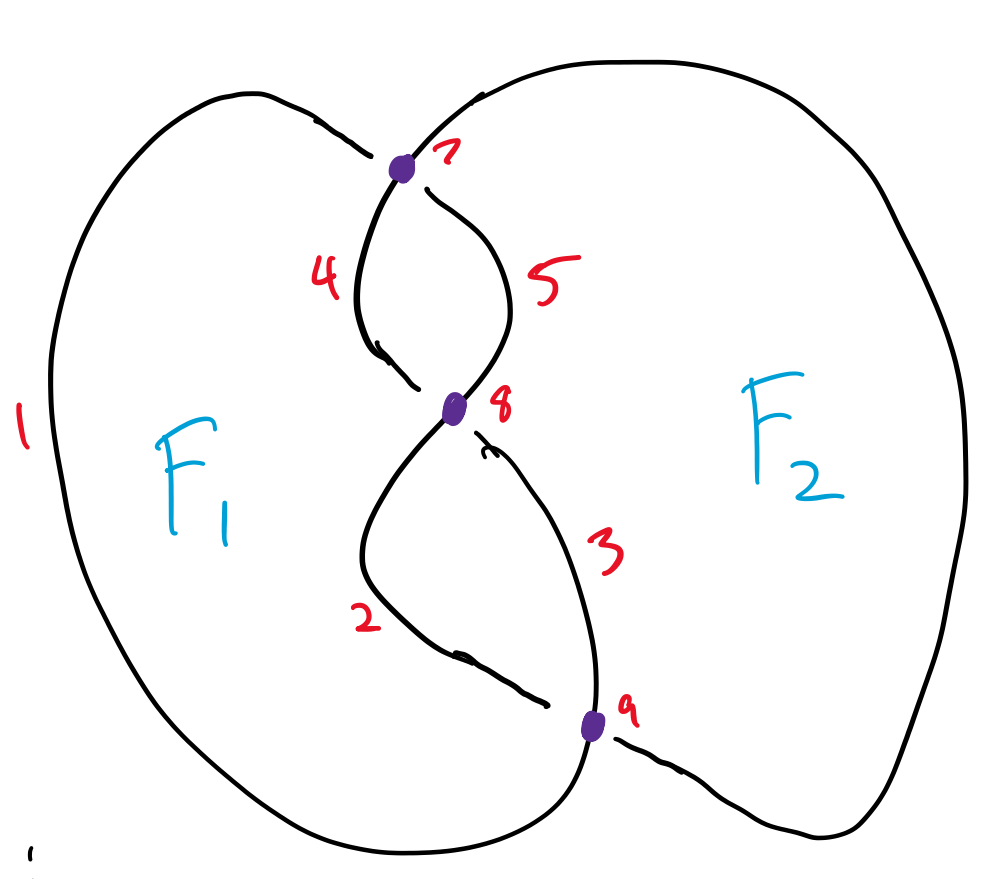
Surface w/ boundary S^1



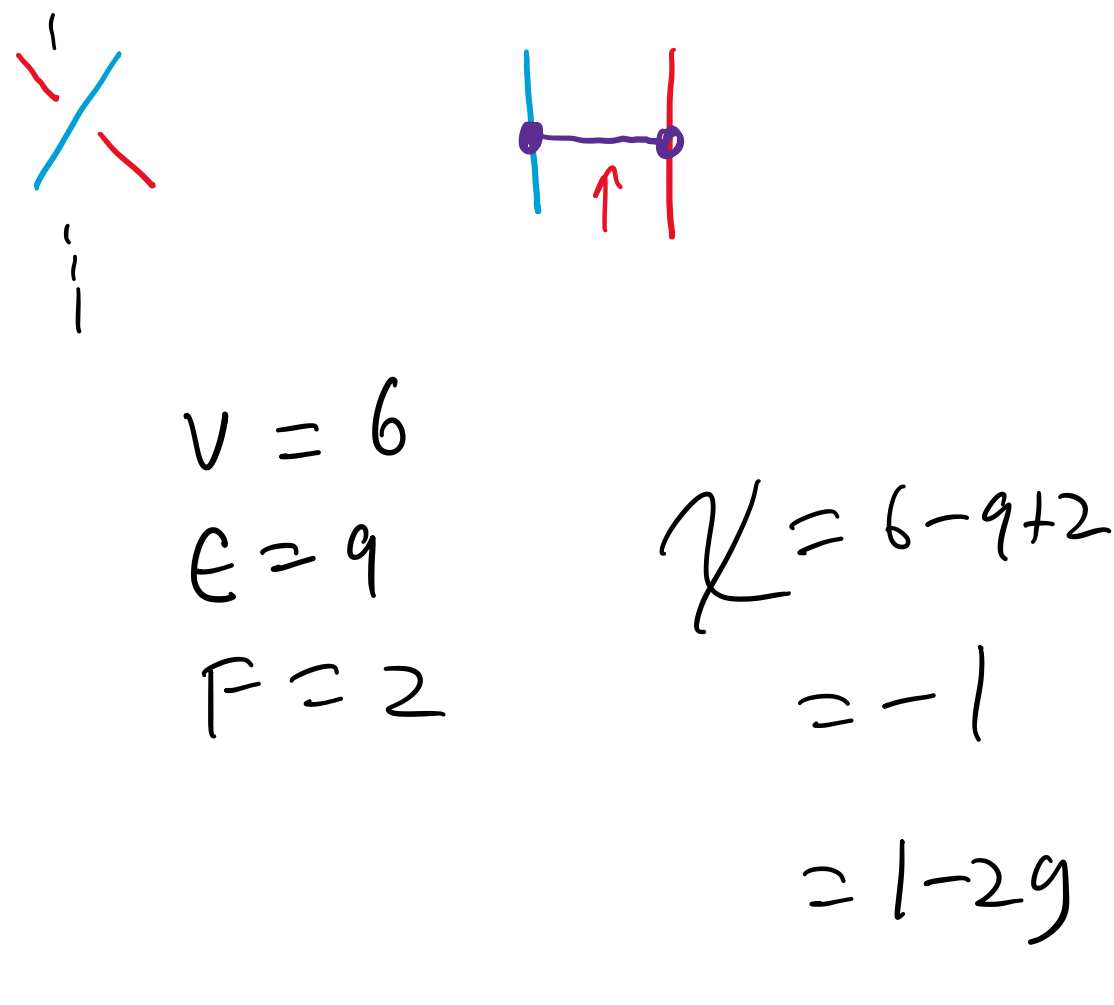
Surface is a Möbius band $\cong \underbrace{\mathbb{R}P^2 - \mathbb{D}^2}_{\text{from our list}}$

Ex]


Is S orientable? Yes

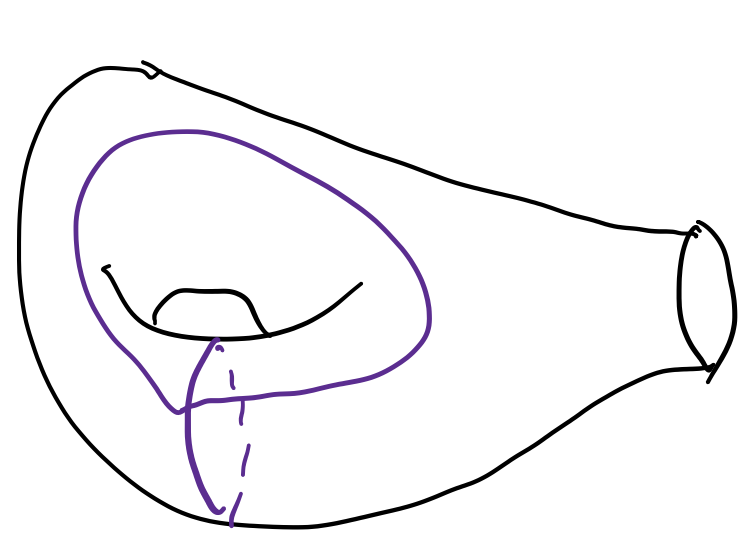


What's the genus?



$V=6$
$E=9$
$F=2$

$\chi = 6-9+2$
$= -1$
$= 1-2g$

$g=1$