Lecture 6

Wednesday, July 2, 2025 12:58 PM

Last time

Spheres

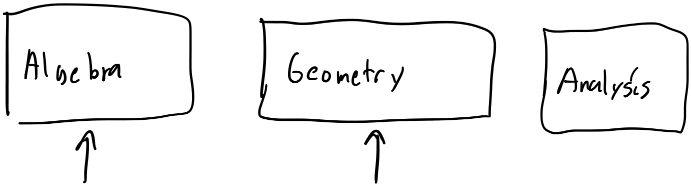
Genus of a knot in S^3

Today

Jones polynomial

Roots in deep algebra

(representation theory of groups)



This is where the Jones poly.

so far, we've been here

comes from.

Polynomials

$3x^2 - 4x + 5$: quadratic polynomial

$6x^4 - 3x^3 + x - 1$: quartic polynomial

$10x^5 - 2x^4 + x^3 - x^2 + 7$: quintic

A general polynomial:

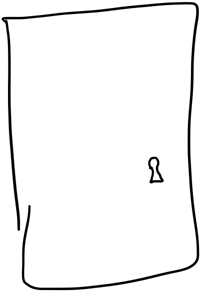
$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

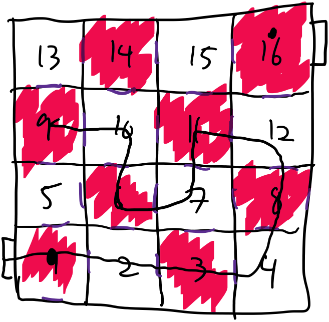
Laurent Polynomials:

Allow exponents to be negative.

Ex] $3x^2 - x + 1 - x^{-1} + 2x^{-3}$

Invariant	Values
Euler char	Integers
Genus of a knot	Naturals (0, 1, 2, ...)
Orientability	{orientable, non-orientable}
Jones Poly.	Laurent polynomials (w/ integer coeffs)





Def: The Kauffman bracket polynomial

$$\langle D \rangle$$

is defined by

1) $\langle O \rangle = 1$

2) $\underline{\langle D \sqcup O \rangle}_{\uparrow} = \underbrace{(-A^{-2} - A^{2})}_{\uparrow} \langle D \rangle$

3) $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

@90°

(3') $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$

Ex] $D = \bigcirc \qquad \langle \bigcirc \rangle = 1$

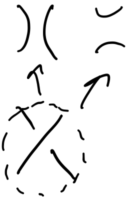
Ex] $D = \overset{\overset{D}{\downarrow}}{\bigcirc} \overset{\sqcup}{} \overset{\overset{\bigcirc}{\downarrow}}{\bigcirc}$

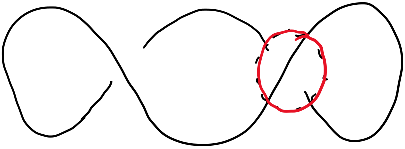
$$\langle D \rangle = (-A^{-2} - A^{2})\langle \bigcirc \rangle$$

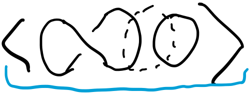
$$= (-A^{-2} - A^{2})$$

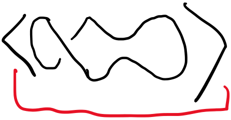
Ex] $D = \underbrace{\bigcirc\ \overset{\downarrow}{\bigcirc}\ \overset{\downarrow}{\bigcirc} \cdots \overset{\downarrow}{\bigcirc}}_{n \text{ components}}$

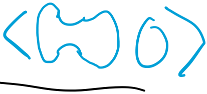
$$\langle D \rangle = \underbrace{(-A^{-2} - A^{2})^{n-1}}$$

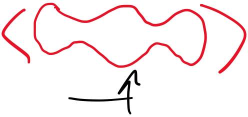
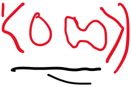


Ex] $D =$ 

$\langle D \rangle = A$ 

$+A^{-1}$ 

$= A \big(A$  $+ A^{-1} \langle \bigcirc\bigcirc\bigcirc \rangle \big)$

$+ A^{-1} \big(A$  $+ A^{-1}$  $\big)$

$$= A^2(-A^{-2}-A^2)$$

$$+ (-A^{-2}-A^2)^2$$

$$+ 1$$

$$+ A^{-2}(-A^{-2}-A^2)$$

$$= \cancel{-1} - \cancel{A^4}$$

$$+ \cancel{A^{-4}} + \cancel{2} + \cancel{A^4}$$

$$+ 1$$

$$\cancel{-1} - \cancel{A^{-4}} = 1$$

For surfaces:

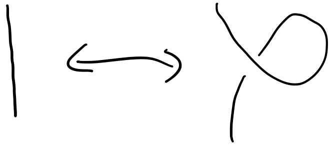
$$\begin{tikzcd}
\text{CW complexes} \arrow[d] & \\
C_1 \arrow[dr] & \\
C_2 \arrow[r] & M \\
C_3 \arrow[ur] &
\end{tikzcd}$$

For knots:

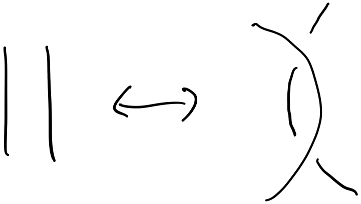
$$\begin{tikzcd}
\text{Diagrams} \arrow[d] & \\
D_1 \arrow[dr] & \\
D_2 \arrow[r] & K \\
D_3 \arrow[ur] &
\end{tikzcd}$$

Reidemeister moves

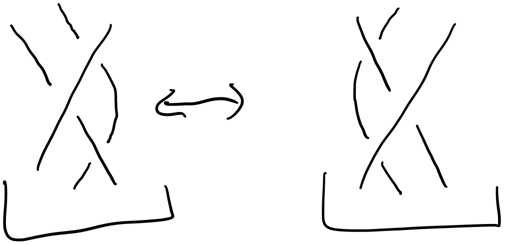
R_1:



R_2:



R_3:

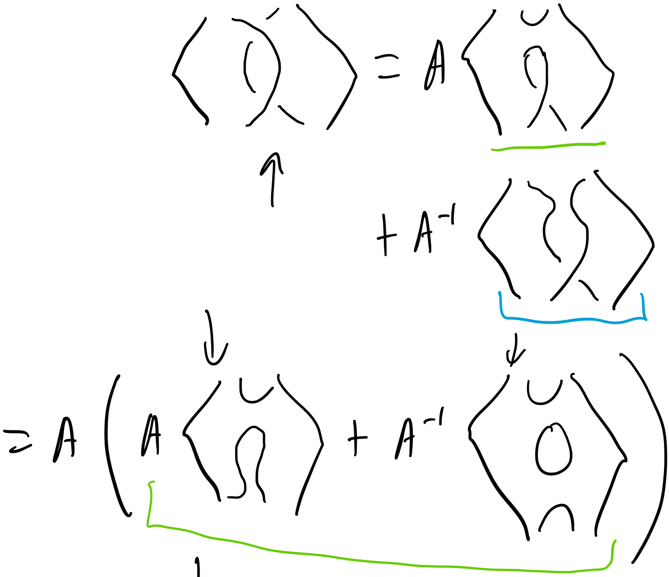


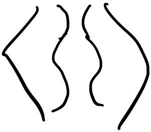
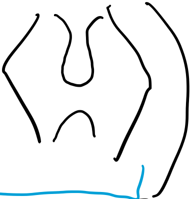
Theorem: If D_1, D_2 are diagrams for a knot K, D_1 and D_2 can be connected by a sequence of Reidemeister moves.

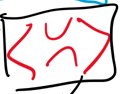
Theorem: If D_1, D_2 differ by an R2 or R3 move,

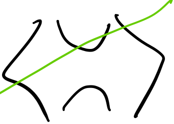
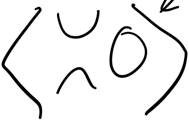
$$\langle D_1 \rangle = \langle D_2 \rangle.$$

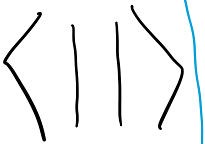
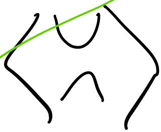
Pf for R2:

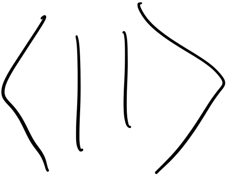


$+ A^{-1} \big(A$  $+ A^{-1}$  $\big)$

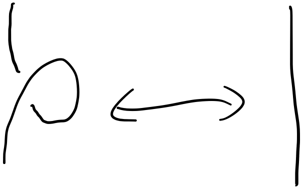
~~$(-A^{-2} - A^{2})$~~ 

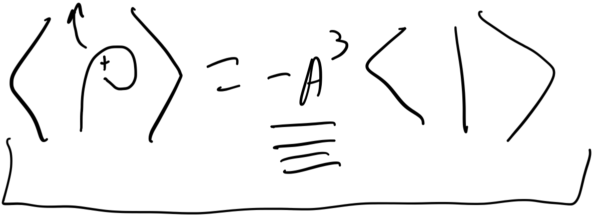
~~$= A^{2}$~~  $+$ 

$+$  ~~$+ A^{-2}$~~ 

$=$  ∎

n-crossing diagram has 2^n resolutions.

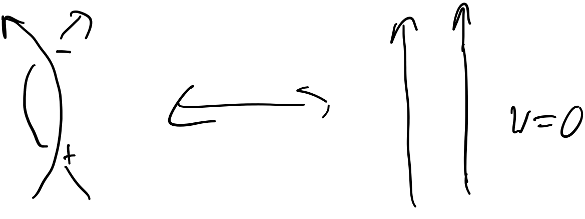
What about R1? 



Def: The writhe $w(D)$ is

positive crossings − # negative crossings

R_2 and R_3 don't change the writhe.



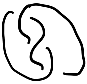
Def: The Jones polynomial $J(D)$ is given by

$$J(D) = (-A^3)^{-w(D)} \langle D \rangle$$

Theorem: The Jones polynomial $J(D)$ is a knot invariant.

right-handed trefoil

Ex] $J(RHT) = -A^{-16} + A^{-12} + A^{-4}$



$t = A^4$

$$= \boxed{\underline{-t^{-4} + t^{-3} + t^{-1}}}$$

Thm: All knots w/ fewer than 10 crossings have distinct Jones polynomials.

Conjecture: If $J(K) = 1$, then K is the unknot.

Thm: $J(m(K))[t] = J(K)[t^{-1}]$

Ex] $J(LHT) = \boxed{t + t^3 - t^4}$

We just proved: $RHT \neq LHT$