Lecture 6 Materials

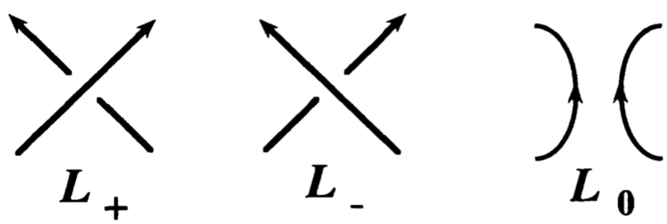
Topology, Knot Theory, and Manifolds

July 2, 2025

Theorem 1. The Jones polynomial satisfies the equation

$$t^{-1}J(L_+) - tJ(L_-) = (t^{-1/2} - t^{1/2})J(L_0)$$

where L_+, L_-, L_0 are diagrams that differ at a crossing as shown below.



Exercise 1.1. A diagram for the Hopf link is shown in Figure 1. (The +1's indicate that the crossings are positive crossings.) Compute the Jones polynomial for the Hopf link directly, i.e. using the Kauffman bracket polynomial.

Exercise 1.2. We're going to compute the Jones polynomial for the Hopf link again, but this time using Theorem 1. After subbing $t = A^4$ (or equivalently, $A = t^{1/4}$), we know from class that $J(O) = 1$ and $J(O\ O) = -t^{-1/2} - t^{1/2}$, where (O) is the unknot and $(O\ O)$ is the two-component unlink.

Note that both crossings are positive crossings. If we pick one, we can call the diagram L_+, and we a new diagram L_- by changing the crossing to a negative crossing. This is a diagram for a two-component unlink (apply Reidemeister II). Similarly, if we change the crossing to the oriented smoothing L_0, we get a diagram for the unknot.

Compute the Jones polynomial of the Hopf link by applying Theorem 1 this crossing. Hint: the Hopf link is L_+ and we already know $J(L_-)$ and $J(L_0)$, so we just need to solve for $J(L_+)$ in this equation.

Exercise 1.3. A diagram for the right-handed trefoil knot is shown in Figure 2. Using the same process as exercise 1.2 (i.e. pick a crossing and apply Theorem 1), compute the Jones polynomial of the right-handed trefoil.

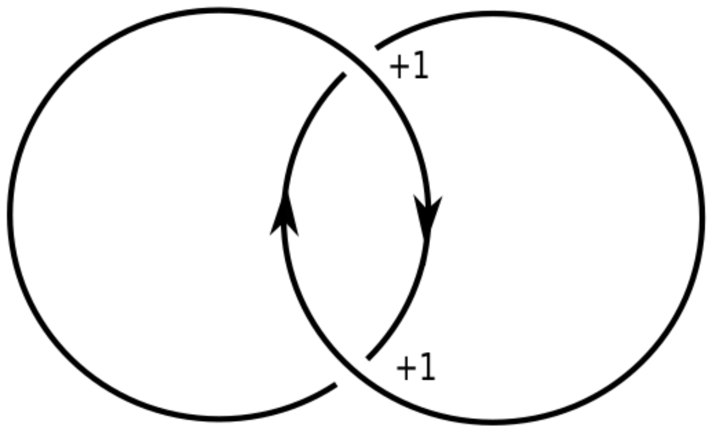


Figure 1: A diagram for the Hopf link.

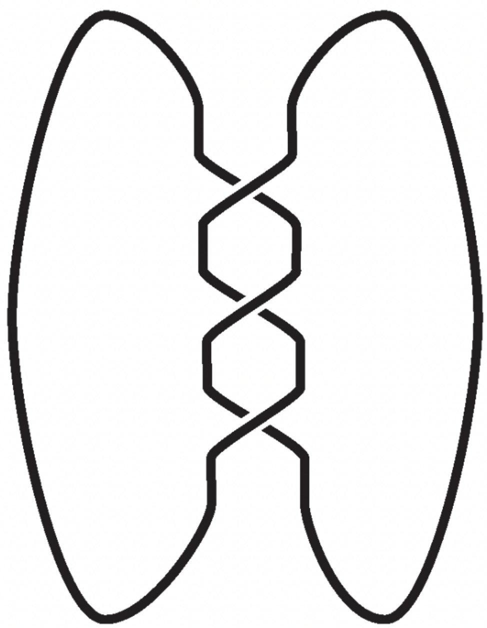


Figure 2: A diagram for the right-handed trefoil knot.

Exercise 1.4. The $(2, n)$ torus knot $T_{2,n}$ is given in Figure 3. Note that $T_{2,1}$ is the unknot, $T_{2,2}$ is the Hopf link, and $T_{2,3}$ is the trefoil. Using Theorem 1, come up with a recursion formula for $J(T_{2,n})$ in terms of $J(T_{2,n-1})$ and $J(T_{2,n-2})$.

One can actually get an explicit formula for $J(T_{2,n})$ from this recursion formula, but it's pretty hard. Something easier to see is that the polynomial keeps getting bigger. This implies that $J(T_{2,n})$ is always different for different values of n, so we just showed that all of the $(2, n)$ torus knots are actually different knots!

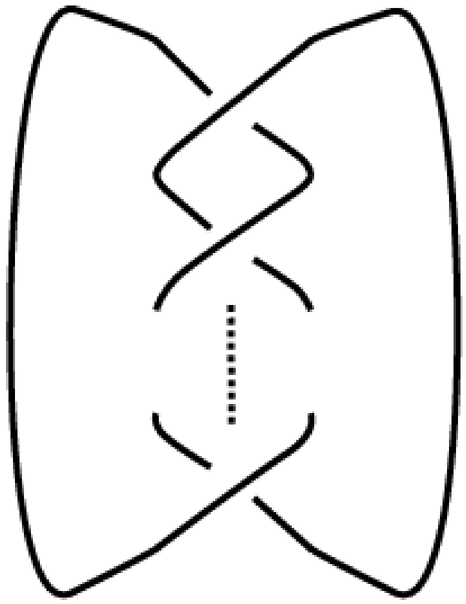


Figure 3: A diagram for the $(2, n)$ torus knot.

Exercise 1.5. A knot K is called *prime* if whenever $K = K_1 + K_2$, either $K_1 = O$ and $K_2 = K$ or $K_1 = K$ and $K_2 = O$ (where again O is the unknot). Show that if $g(K) = 1$, then K is prime.

Outside Content 1.6. A paper by Jones himself introducing the Jones polynomial.