Lecture 7

Monday, July 7, 2025 12:59 PM

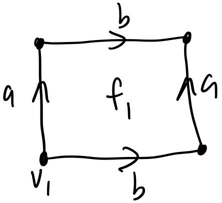
Last Time

Jones Polynomial

Today

Homology (Categorification)

Ex]



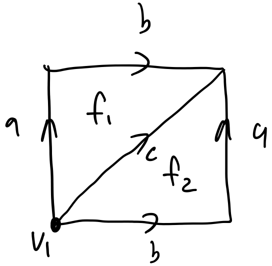
$V = 1$

$E = 2$

$F = 1$

$\chi = F - E + V$

$= 1 - 2 + 1 = \underline{\underline{0}}$



$V = 1$

$E = 3$

$F = 2$

$\chi = 2 - 3 + 1 = \underline{\underline{0}}$

$f: T^2 \longrightarrow S^2$

$\chi: 0 \cdots\cdots 2$

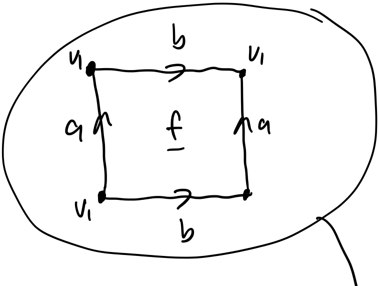
$\mathbb{F} = \mathbb{Z}_2 = \mathbb{Z}/_{2=0} = \{0, 1\}$

$1+1=2=0$

CW = a CW complex

Homology

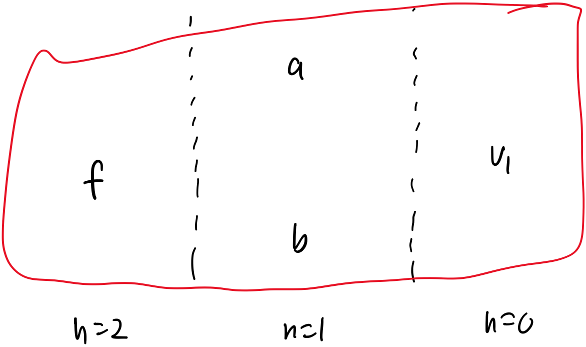
$C_n(CW) = \mathbb{F}^k$ where k is the number of n-dim cells



$C_2 = \mathbb{F} \quad (f)$

$C_1 = \mathbb{F}^2 \quad (a, b)$

$C_0 = \mathbb{F} \quad (v_1)$



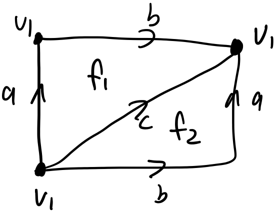
$\partial : C_n \longrightarrow C_{n-1}$ maps an n-cell to the sum of its boundary cells.

$$\partial f = a + b + a + b$$
$$= 2a + 2b = 0$$

$$\partial a = v_1 + v_1 = 2v_1 = 0$$
$$\partial b = v_1 + v_1 = 2v_1 = 0$$

$\partial v_1 = 0$

Ex]



$\partial f_1 = a + b + c$

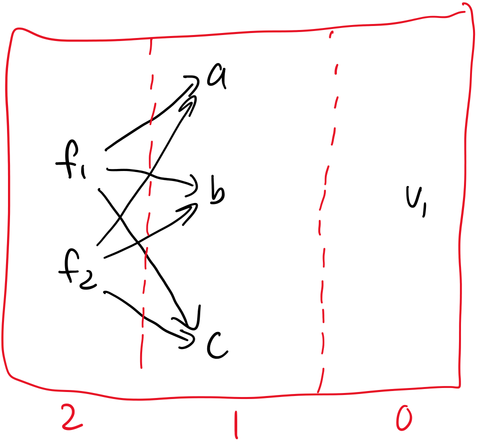
$\partial f_2 = a + b + c$

$\partial a = v_1 + v_1 = 0$

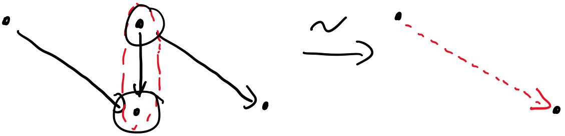
$\partial b = 0$

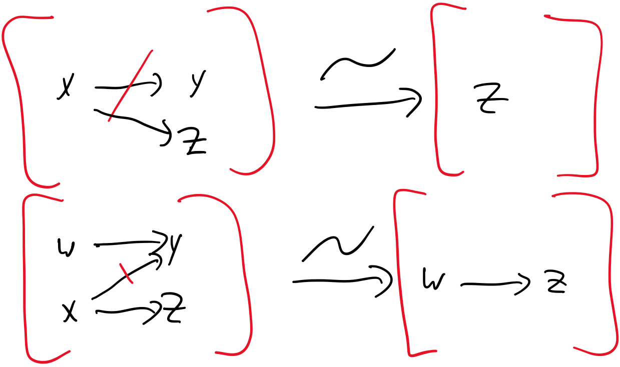
$\partial c = 0$

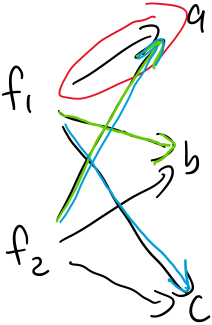
$\partial v_1 = 0$



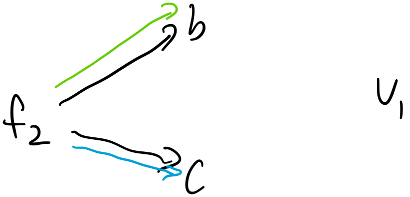
Cancelation Law



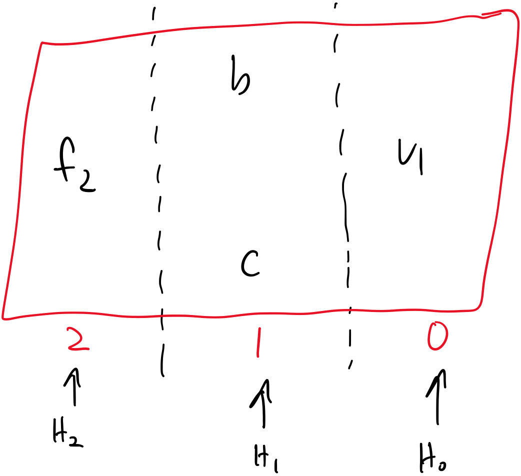




v_1



$2 = 0$



$$H_2(T^2) = \mathbb{F}^1$$
$$H_1(T^2) = \mathbb{F}^2$$
$$H_0(T^2) = \mathbb{F}^1$$

$\mathbb{F}^n \sim n$ generators
(dots, letters)

$$\text{rank}(\mathbb{F}^n) = n$$

$B_k = \text{rank}(H_k)$ (k-th betti number)

$B_2(T^2) = 1$

$B_1(T^2) = 2$

$B_0(T^2) = 1$

Def: $\chi(m) = \sum_n (-1)^n \text{rank}(C_n(CW))$

$(= V - E + F)$

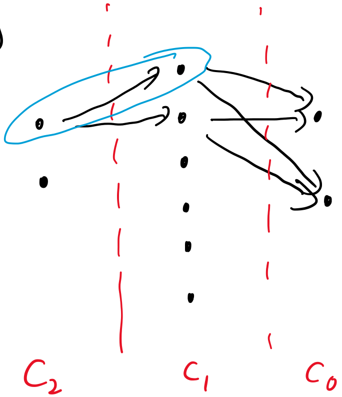
Thm: $\sum_n (-1)^n \text{rank}(H_n(m)) = \chi(m)$

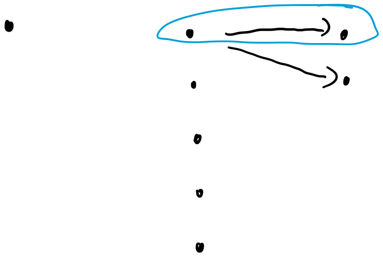
Pf: Cancellation law preserves alternating sum.

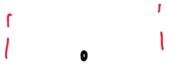
Thm: $H_i(m)$ is an invariant of

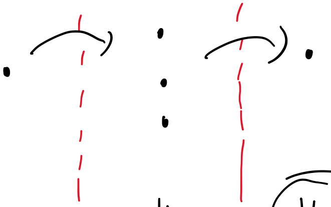
m for all i.

Ex]









H_2 $\qquad$ H_1 $\qquad$ $\boxed{H_0}$

$B_2=1$ $\qquad$ $B_1=4$ $\qquad$ $B_0=1$

$$\chi = -2 \Rightarrow M = \begin{matrix} T^2 \# T^2 \\ \text{or } \#^4 \mathbb{R}P^2 \end{matrix}$$

Theorem: B_0 is the number of connected components of M.

Theorem (Poincaré duality): For an oriented n-dim. manifold,

$$\underbrace{B_i = B_{n-i}}_{}$$

Ex]

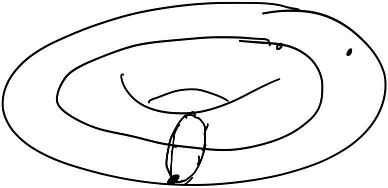
B_i

m	0	1	2	3
S^3	1	0	0	1
$S^1 \times S^2$	1	1	1	1
$S^1 \times S^1 \times S^1$ (T^3)	1	3	3	1

$\chi = 0$ for all orientable 3-manifolds.

$T^2 = S^1 \times S^1$ (a, b)

$a \in S^1$
$b \in S^1$



$$\left.\begin{array}{l} H_0(T^2) = \mathbb{F} \\ H_1(T^2) = \mathbb{F}^2 \\ H_2(T^2) = \mathbb{F} \end{array}\right\}$$ homology of the torus.

Back to Knot Theory

$J(RHT) = 1t^1 + 1t^3 - 1t^4$

	∂	$x=1$ ↓	$x=0$ ↓	$x=1$ ↓	$x=-1$ ↓
3	↓	0	0	0	1
2	↓ 0	0	0	1	0
1	↓ 0	0	0	0	0
$h=0$	0	1	0	0	0
	0	0	0	0	0
	$t=0$	1	2	3	4

↑

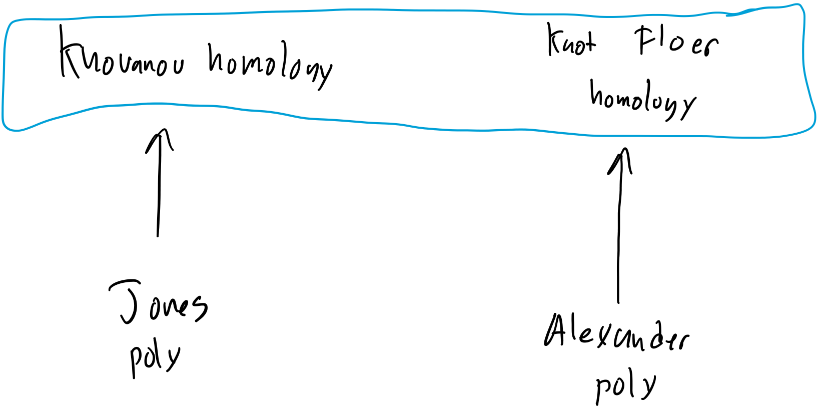
Khovanov homology

χ (Khovanov homology) = Jones polynomial

Conj: If $J(K) = 1$, then K is the unknot.

Theorem: If the total rank of $Kh(K)$ is 1, then K is the unknot.

Khovanov homology also detects the trefoils, the figure-eight knot, and a few others.



Def: The crossing number $c(K)$ is the smallest number of crossings in any diagram for K.

Q: What is $c(7_4)$?

A: 7

Conjecture: $c(K_1 + K_2) = c(K_1) + c(K_2)$

Def: The unknotting number $u(K)$ is the fewest times a knot must pass through itself to become unknotted.

Conjecture: $u(K_1 + K_2) = u(K_1) + u(K_2)$