Lecture 8

Wednesday, July 9, 2025 12:33 PM

So far

Manifold invariants

Orientability

Euler characteristic

Homology

Genus (2D only)

Knot invariants

Genus

Jones polynomial

Crossing number } properties

Unknotting number } properties

Khovanov homology

Alexander polynomial

Knot Floer homology

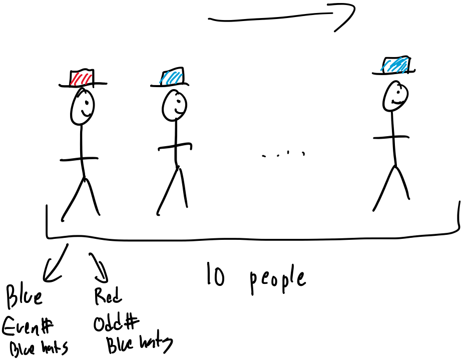
Today

4-ball genus

Movie moves

Concordance Group

Riddle



Each person guesses the color of their own hat, from back to front.

Need: 9/10 get it right.

Hint: $\boxed{\mathbb{Z}_2}$

Q: What if ther are 100 colors?

Same strategy, but over $\mathbb{Z}/100$

Q: What if there are ∞ colors?

Map the colors onto $[0, 1)$,
then look mod 1.

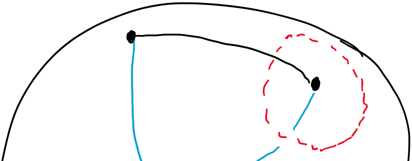
Q: What is $\mathbb{R}/1=0$ topologically?

$$\mathbb{R}/_{x=x+1} = S^1$$

Knots: $S^1 \hookrightarrow S^3$ $\qquad S^3 = \partial B^4$

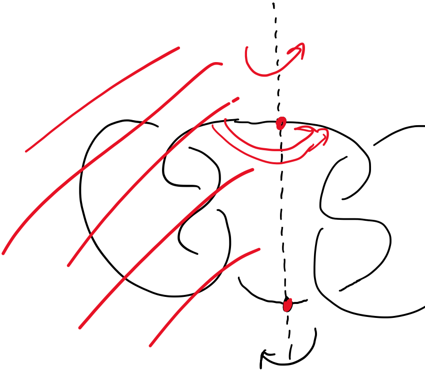
$$g(K) = \min\left(\{g(F) \mid F \text{ is an O.S. in } S^3 \text{ w/ } \partial F = K\}\right)$$

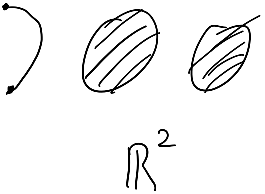
Def: $g_4(K) = \min\left(\{g(F) \mid F \text{ is an O.S. in } B^4 \text{ w/ } \partial F = K\}\right)$



Theorem: $g_4(K) \leq g(K)$

Q: If $g_4(K)=0$, is K an unknot?





$RHT + LHT = K$

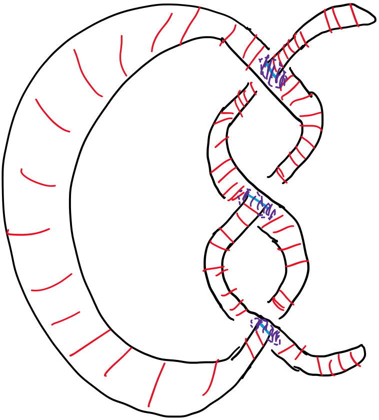
$g=1$ $g=1$ $g=2$

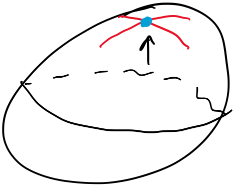
Swing the arc through B^4. Carves out a disc $\mathbb{D}$, $\partial \mathbb{D} = K$.

↑

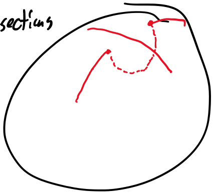
slice disc

RHT + LHT





Pop purple discs into the 4-ball so that there are no self-intersections

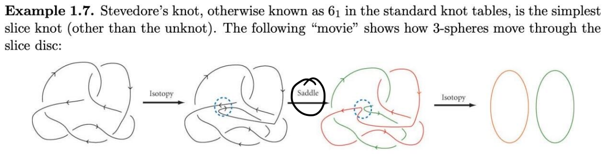


More generally,

$$g_4(K + m(K)) = 0 \quad \text{for all } K.$$

Q: What about prime knots?

$g_4(6_1) = 0$


Isotopy
Saddle
Isotopy

Example 1.7. Stevedore's knot, otherwise known as 6_1 in the standard knot tables, is the simplest slice knot (other than the unknot). The following "movie" shows how 3-spheres move through the slice disc:

6_1

two unknots

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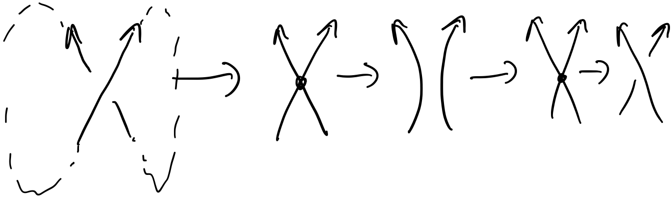
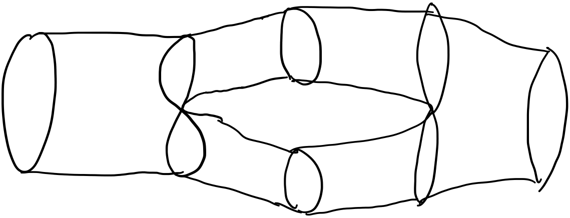
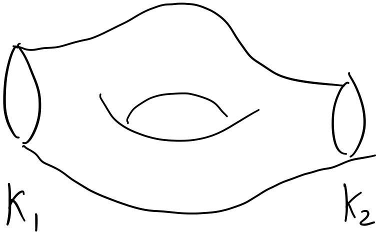
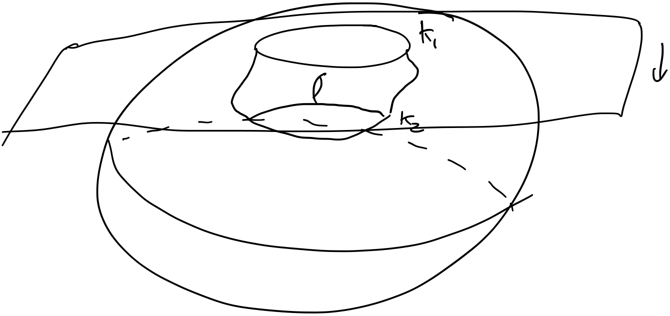
g_4 and unknotting number

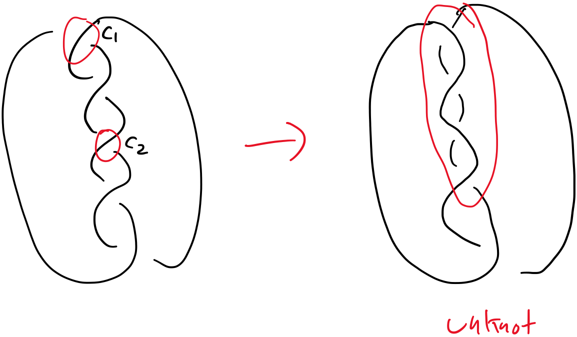


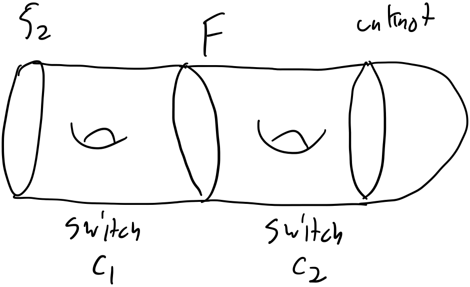






S_2





$$\partial F = S_2$$

$$g(F) = 2$$

$$\boxed{g_4(S_2) \leq 2}$$

Theorem: $g_4(K) \leq u(K)$ for all K.

Slice invariants

Khovanov homology ⟶ s-invariant

Knot Floer homology ⟶ τ-invariant

$$s: \text{Knots} \longrightarrow \mathbb{Z}$$

$$\underline{|s(K)| \leq g_4(K)}$$

$$s(S_2) = 2$$

$$\Rightarrow g_4(S_2) = 2$$

$$\boxed{s(K_1 + K_2) = s(K_1) + s(K_2)}$$

Q: what is s(figure-eight knot)?

$$\boxed{4_1 = m(4_1)}$$

$$g_4(4_1 + m(4_1)) = 0$$

$$g_4(4_1 + 4_1) = 0$$

$$s(4_1 + 4_1) = 0$$

$$s(4_1) + s(4_1) = 0$$

$$s(4_1) = 0.$$

Thm: If K is amphichyral (i.e. $K = m(K)$), then

$$g_4(\underset{\substack{\uparrow \\ 2K}}{K+K}) = 0.$$

Open question: are there numbers n s.t.

$g_4(nK) = 0$ but

$g_4(mK) \neq 0$ for $m < n$.

$\boxed{n=2}$ works. $n=3$?

$n=4$?

Unknown.